1. Find the solution of the given initial value problem. Show your work. (6 points)

$$y' - 2y = e^{2t}, \qquad y(0) = 2.$$

$$\mu(t) = e^{-\int 2\,dt} = e^{-2t}$$

$$y'e^{-2t} - 2ye^{-2t} = e^{2t} \cdot e^{-2t} = 1$$

$$\Rightarrow \left(ye^{-2t}\right)' = 1$$

$$\Rightarrow \quad ye^{-2t} = t + C$$

$y = 2$ when $t = 0$

$$2 = C$$

Therefore $ye^{-2t} = t + 2$

$$\boxed{y = te^{2t} + 2e^{2t}}$$

2. Determine the order of the given differential equations. Also state whether the equation is linear of nonlinear. Explain very briefly the justification of your answer. (6 points)

$$(i)\ \ y''' + 3y'' + 4yy' - y + 2 = 0, \qquad (ii)\ \ u_{xy} + u_{yy} + uu_x = 0$$

(i) 3rd order → presence of y'''

Non-linear → presence of yy'

(ii) 2nd order → presence of u_{xy}

Non-linear → presence of uu_x.

3. Show that the given equation is not exact but becomes exact when multiplied by the given integrating factor. Then solve the equation. Show your work. (8 points)

$$y\,dx + (2x - ye^y)\,dy = 0, \qquad \mu(x, y) = y.$$

$M = y \quad , \quad N = 2x - ye^y$

$M_y = 1 \quad , \quad N_x = 2$

$M_y \neq N_x$. Therefore it's not exact.

After multiplying by y we have

$y^2 dx + (2xy - y^2 e^y)\,dy = 0$

$M = y^2 \quad , \quad N = 2xy - y^2 e^y$

$M_y = 2y \quad , \quad N_x = 2y$

$M_y = N_x$. Therefore exact.

There exists a function Ψ such that $\Psi_x = y^2$ and $\Psi_y = 2xy - y^2 e^y$

$\Psi_x = y^2$

$\Psi = y^2 x + h(y)$

$\Psi_y = 2yx + h'(y)$.

Therefore $2yx + h'(y) = 2xy - y^2 e^y$

$h'(y) = -y^2 e^y$

$h(y) = -\int y^2 e^y\,dy$

$u = y^2$, $dv = e^y\,dy$ $\quad = -\left[y^2 e^y - \int e^y\, 2y\,dy\right]$

$du = 2y\,dy$, $v = e^y$ $\quad = -\left[y^2 e^y - 2\left(ye^y - \int e^y\,dy\right)\right]$

$u = y$, $dv = e^y\,dy$

$du = dy$, $v = e^y$ $\quad = -y^2 e^y + 2ye^y - 2e^y$

Therefore the solution is $\boxed{y^2 x - y^2 e^y + 2ye^y - 2e^y = C}$

4. Find the solution of the given initial value problem in explicit form, and determine the interval in which the solution is defined. Show your work. (8 points)

$$y' = \frac{1-2x}{y}, \qquad y(1) = -2.$$

$\longrightarrow y \neq 0$

$$\frac{dy}{dx} = \frac{1-2x}{y}$$

$$\int y\,dy = \int (1-2x)\,dx$$

$$\frac{y^2}{2} = x - x^2 + C$$

$$\frac{(-2)^2}{2} = 1 - 1^2 + C$$

$$2 = C$$

Therefore

$$\frac{y^2}{2} = x - x^2 + 2$$

$$y^2 = 2x - 2x^2 + 4$$

$$y = \pm\sqrt{2x-2x^2+4}$$

However checking the initial conditions

$$-2 = -\sqrt{2-2+4}$$

$$\therefore \boxed{y = -\sqrt{2x-2x^2+4}}$$

$$2x - 2x^2 + 4 > 0$$

$$x^2 - x - 2 < 0$$

$$(x-2)(x+1) < 0$$

$$\boxed{-1 < x < 2}$$

5. Find the solution of the given initial value problem and describe its behavior for increasing t. Show your work. (6 points)

$$y'' + 4y' + 5y = 0, \qquad y(0) = 1, \quad y'(0) = 0$$

$$r^2 + 4r + 5 = 0$$

$$r = \frac{-4 \pm \sqrt{16-20}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$= -2 \pm i$$

$\lambda = -2$

$\mu = 1$

$$y = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t$$

$$y' = -2c_1 e^{-2t}\cos t - c_1 e^{-2t}\sin t - 2c_2 e^{-2t}\sin t + c_2 e^{-2t}\cos t.$$

$$1 = c_1$$

$$0 = -2c_1 + c_2$$

$$c_2 = 2$$

$$\therefore \boxed{y = e^{-2t}\cos t + 2e^{-2t}\sin t}$$

$\boxed{\text{As } t \to \infty \quad y \to 0, \text{ decaying oscillations}}$

6. Verify that the functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions? Give a reason by showing your work. (8 points)

$$x^2y'' - x(x+2)y' + (x+2)y = 0, \quad x > 0; \quad y_1(x) = x, \quad y_2(x) = xe^x$$

$y_1 = x$

$y_1' = 1$

$y_1'' = 0$

$x^2 . 0 - x(x+2).1 + (x+2)x$

$= -x(x+2) + x(x+2)$

$= 0 = RHS \quad \therefore y_1$ is a solution.

$y_2 = xe^x$

$y_2' = xe^x + e^x$

$y_2'' = xe^x + e^x + e^x$

$= xe^x + 2e^x$

$x^2(xe^x + 2e^x) - x(x+2)(xe^x + e^x) + (x+2)xe^x$

$x^2e^x(x+2) - x(x+2)e^x(x+1) + (x+2)xe^x$

$(x+2)e^x x(x+1) - x(x+2)e^x(x+1)$

$= 0 = RHS \quad \therefore y_2$ is a solution.

$$W(y_1, y_2) = \begin{vmatrix} x & xe^x \\ 1 & xe^x + e^x \end{vmatrix}$$

$= x(xe^x + e^x) - xe^x . 1$

$= x^2e^x + \cancel{xe^x} - \cancel{xe^x}$

$= x^2e^x \neq 0 \quad \therefore y_1$ and y_2 form a fundamental set of solutions.

7. Use the Method of Undetermined Coefficients to find the particular solution of the given differential equation, then find its **general solution**. Show your work. (8 points)

$$y'' + 4y' + 4y = 6\sin 2t$$

$Y = A\cos 2t + B\sin 2t$

$Y' = -2A\sin 2t + 2B\cos 2t$

$Y'' = -4A\cos 2t - 4B\sin 2t$

$-4A\cos 2t - 4B\sin 2t + 4\left(-2A\sin 2t + 2B\cos 2t\right) + 4\left(A\cos 2t + B\sin 2t\right) = 6\sin 2t$

$\Rightarrow -\cancel{4A}\cos 2t - \cancel{4B\sin 2t} - 8A\sin 2t + 8B\cos 2t + \cancel{4A\cos 2t} + \cancel{4B\sin 2t} = 6\sin 2t$

$\Rightarrow -8A\sin 2t + 8B\cos 2t = 6\sin 2t$

$8B = 0 \Rightarrow B = 0$

$-8A = 6$

$A = -\frac{3}{4}$

$\therefore \boxed{Y(t) = -\frac{3}{4}\cos 2t}$ Particular solution.

<u>Homogeneous equation</u>

$y'' + 4y' + 4y = 0$

$r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0 \Rightarrow r = -2, -2$

$\therefore y = c_1 e^{-2t} + c_2 t e^{-2t}$

General solution: $\boxed{y = c_1 e^{-2t} + c_2 t e^{-2t} - \frac{3}{4}\cos 2t}$