Manual for MATH 3860, Spring 2012

Instructor : Dibyajyoti Deb

Disclaimer: This manual is in no way a short cut to learning differential equations. You will still need to understand the theory from the text book to succeed in the exams. What I have in this manual is a collection of all the different techniques that we have learnt in this course to just solve differential equations.

Throughout this manual t and x will be the independent variable and $y^{(n)} = \dfrac{d^n y}{dt^n}$ and $y^{(n)} = \dfrac{d^n y}{dx^n}$ respectively.

Contents

1.1. First Order Linear Equations

1.1.1. General Form

$$P(t)y' + Q(t)y = G(t). \tag{1.1}$$

1.1.2. How to solve it?

- Write the general form as

$$y' + p(t)y = g(t), \tag{1.2}$$

  by dividing both sides by $P(t)$.
- Find the integrating factor

$$\mu(t) = e^{\int p(t)\,dt}$$

- Multiply both sides of Eq. (1.2) with $\mu(t)$

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t),$$

  Left side becomes,

$$(\mu(t)y)' = \mu(t)g(t)$$

- Integrate both sides to get,

$$\mu(t)y = \int \mu(t)g(t)\,dt + C,$$

  where C is an arbitrary constant which can be determined if there is an initial condition.

1.1.3. An example

Check **Example 1** from **Lecture 2**.

1.2. Separable Equations

1.2.1. General Form

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{1.3}$$

Here M is a function of the independent variable x and N is a function of y.

1.2.2. How to solve it?

- Write the general form as

$$M(x)\,dx + N(y)\,dy = 0, \tag{1.4}$$

- Integrate both sides and write the final constant(from both integrals) on the right side.

$$\int M(x)\,dx + \int N(y)\,dy = C,$$

where C is an arbitrary constant which can be determined if there is an initial condition.

1.2.3. An example

Check **Example 2** from **Lecture 2**.

1.3. Exact Equations

1.3.1. General Form

Exact equations are usually neither separable nor linear. Their general form is

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

which satisfies

$$M_y(x,y) = N_x(x,y) \tag{1.5}$$

This also means that there exists a function ψ such that,

$$\psi_x(x,y) = M(x,y) \quad \psi_y(x,y) = N(x,y). \tag{1.6}$$

1.3.2. How to solve it?

- Check (1.5) to verify that the given equation is exact.
- Therefore

$$\psi_x(x,y) = M(x,y) \tag{1.7}$$

and

$$\psi_y(x,y) = N(x,y) \tag{1.8}$$

for some function ψ.
- Integrate (1.7) (with respect to x) to get

$$\psi(x,y) = \int M(x,y)\,dx + h(y), \tag{1.9}$$

- Differentiate both sides with respect to y to get

$$\psi_y(x,y) = \frac{\partial Q(x,y)}{\partial y} + h'(y) \tag{1.10}$$

where $Q(x,y) = \displaystyle\int M(x,y)\,dx$.

- Therefore

$$\frac{\partial Q(x,y)}{\partial y} + h'(y) = N(x,y) (\text{since both are equal to } \psi_y(x,y)) \tag{1.11}$$

- Therefore

$$h'(y) = N(x,y) - \frac{\partial Q(x,y)}{\partial y} \tag{1.12}$$

which should be a function of y only. $h(y)$ can be found by integrating both sides of Eq. (1.12).

$$h(y) = \int \left(N(x,y) - \frac{\partial Q(x,y)}{\partial y}\right) dy \tag{1.13}$$

- So substituting $h(y)$ back in Eq. (1.9) we have

$$\psi(x,y) = \int M(x,y)\,dx + \int \left(N(x,y) - \frac{\partial Q(x,y)}{\partial y}\right) dy \tag{1.14}$$

The solution of the differential equation is of the form

$$\psi(x,y) = C. \tag{1.15}$$

for any arbitrary constant C.

1.3.3. An example

Check **Example 3** from **Lecture 3**.

1.4. Non exact to Exact equation

1.4.1. General Form

Sometimes it's possible to convert an equation that is not exact into an exact equation by multiplying with an integrating factor $\mu(x,y)$.

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0, \tag{1.16}$$

however,

$$M_y(x,y) \neq N_x(x,y) \tag{1.17}$$

1.4.2. How to solve it?

- Find

$$Q = \frac{M_y - N_x}{N} \quad \text{and} \quad R = \frac{N_x - M_y}{M} \tag{1.18}$$

- If Q is a function of only x, then set

$$\frac{d\mu}{dx} = Q(x)\mu \tag{1.19}$$

  and solve for μ (it's a separable equation).
- If R is a function of only y, then set

$$\frac{d\mu}{dy} = R(y)\mu \tag{1.20}$$

  and solve for μ (it's a separable equation).
- Note that either Q being a function of only x or R being a function of only y is enough to find μ (don't need both at the same time).
- After finding μ, multiply both sides of Eq. (1.16) by μ

$$\mu M(x, y) + \mu N(x, y)\frac{dy}{dx} = 0 \tag{1.21}$$

  which will make it an exact equation and earlier methods can be used to solve it.

1.4.3. An example

Check **Example 4** from **Lecture 3**.

1.5. Second order linear homogeneous equations with constant coefficients

1.5.1. General Form

$$ay'' + by' + cy = 0, \tag{1.22}$$

where a, b and c are constants.

1.5.2. How to solve it?

- Form the characteristic equation

$$ar^2 + br + c = 0 \tag{1.23}$$

- Solve Eq. (1.23) to find the two roots r_1 and r_2.
  - If r_1 and r_2 are real and $r_1 \neq r_2$, the general solution is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{1.24}$$

– If r_1 and r_2 are real and $r_1 = r_2$, the general solution is

$$y = c_1 e^{r_1 t} + c_2 t e^{r_1 t} \tag{1.25}$$

– If r_1 and r_2 are complex then let $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, the general solution is

$$y = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t \tag{1.26}$$

for arbitrary constants c_1 and c_2 which can be determined with initial conditions.

1.5.3. An example

Check **Example 2** from **Lecture 4**.

1.6. Second order linear homogeneous equation: Reduction of Order

1.6.1. General Form

$$y'' + p(t)y' + q(t)y = 0 \tag{1.27}$$

where one solution $y_1(t)$ is already known.

1.6.2. How to solve it?

- Set

$$y = v(t)y_1(t) \tag{1.28}$$

Find

$$y' = v'(t)y_1(t) + v(t)y_1'(t). \tag{1.29}$$

and

$$y'' = v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t). \tag{1.30}$$

- Substitute y, y' and y'' back into Eq. (1.27) and collect the terms to make a differential equation in v

$$y_1 v'' + (2y_1' + py_1)v' + (y_1'' + py_1' + qy_1)v = 0. \tag{1.31}$$

- The coefficient of v in Eq. (1.31) is zero since y_1 is a solution of Eq. (1.27), so Eq. (1.31) becomes

$$y_1 v'' + (2y_1' + py_1)v' = 0. \tag{1.32}$$

- Think of v'' as $(v')'$ and Eq. (1.32) then becomes a linear equation(also a separable equation) in v'. Solve it to find v' and then v. Substitute v back into Eq. (1.28) to find the other solution.

1.6.3. An example

Check **Example 2** from **Lecture 6**.

1.7. nth order Linear homogeneous equations with constant coefficients

1.7.1. General Form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0 \tag{1.33}$$

where $a_0, a_1, \ldots, a_n$ are real constants.

1.7.2. How to solve it?

- Form the characteristic equation

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0 \tag{1.34}$$

- Solve Eq. (1.34) to find n roots $r_1, r_2, \ldots, r_n$.

Unequal roots : If $r_1, r_2, \ldots, r_m$ are real and $r_1 \neq r_2 \neq \cdots \neq r_m$, where $m \leq n$, then the solutions that come out these roots are

$$e^{r_1 t}, \quad e^{r_2 t}, \quad \ldots, \quad e^{r_m t} \tag{1.35}$$

Complex roots : If a root r_1 is complex then there exists another root r_2 such that $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$ (complex roots occurs in conjugate pairs), the solutions that come out of these two roots are

$$e^{\lambda t} \cos \mu t, \qquad e^{\lambda t} \sin \mu t \tag{1.36}$$

Repeated roots : If a root r_1 is real and repeated s times, then the s solutions that comes out of these repeated roots are

$$e^{r_1 t}, \quad t e^{r_1 t}, \quad t^2 e^{r_1 t}, \quad \ldots, \quad t^{s-1} e^{r_1 t} \tag{1.37}$$

If a root $r_1 = \lambda + i\mu$ is complex and repeated s times, then its complex conjugate $\lambda + i\mu$ is also repeated s times. The $2s$ solutions that come out of these roots are

$$e^{\lambda t} \cos \mu t, \quad e^{\lambda t} \sin \mu t, \quad t e^{\lambda t} \cos \mu t, \quad t e^{\lambda t} \sin \mu t,$$
$$\ldots, \quad t^{s-1} e^{\lambda t} \cos \mu t, \quad t^{s-1} e^{\lambda t} \sin \mu t.$$

- Once all the solutions $y_1, y_2, \ldots, y_n$ have been found corresponding to all the roots of the characteristic equations, the general solution of Eq. (1.33) is given by

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) \tag{1.38}$$

for arbitrary constants $c_1, c_2, \ldots, c_n$ which can be determined with initial conditions.

1.7.3. An example

Check **Example 2** from **Lecture 7**.

1.8. nth order Linear nonhomogeneous equations with constant coefficients: Method of Undetermined Coefficients

1.8.1. General Form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t) \tag{1.39}$$

where $a_0, a_1, \ldots, a_n$ are constants.

1.8.2. How to solve it?

- Construct the corresponding homogeneous equation
$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0 \tag{1.40}$$
and solve it. Let the solutions of this homogeneous equation be $y_1, y_2, \ldots, y_n$.
- Make sure that the function $g(t)$ in Eq. (1.39) belongs to one of the classes of functions in the next table, that is, it involves nothing more than exponential functions, sines, cosines, polynomials, or sum or products of such functions.
- If $g(t) = g_1(t) + \cdots + g_n(t)$, that is, if $g(t)$ is a sum of n terms, then form n subproblems, each of which contains only one of the terms $g_1(t), \ldots, g_n(t)$. The ith subproblem consists of the equation
$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g_i(t), \tag{1.41}$$
where i runs from 1 to n.
- Depending on $g_i(t)$, we assume the particular solution $Y_i(t)$ according to the next table.

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n$	$A_0 t^n + A_1 t^{n-1} + \cdots + A_n$
$P_n(t) e^{\alpha t}$	$(A_0 t^n + A_1 t^{n-1} + \cdots + A_n) e^{\alpha t}$
$P_n(t) e^{\alpha t} \sin \beta t$ or $P_n(t) e^{\alpha t} \cos \beta t$	$(A_0 t^n + A_1 t^{n-1} + \cdots + A_n) e^{\alpha t} \cos \beta t + (B_0 t^n + B_1 t^{n-1} + \cdots + B_n) e^{\alpha t} \sin \beta t$

- If there is any duplication in the assumed form of $Y_i(t)$ with the solutions of the corresponding homogeneous equation, then multiply $Y_i(t)$ by t^s, if $Y_i, tY_i, \ldots, t^{s-1}Y_i$ are all solutions of the corresponding homogeneous equation, so as to remove the duplication. So for instance if we want to find a particular solution of

$$y'' + 4y' + 4y = 6te^{-2t}, \tag{1.42}$$

our choice of $Y(t)$ would have to be At^2e^{-2t} since te^{-2t} (which we find from the above table) is a solution of the corresponding homogeneous equation of Eq. (1.42).
- Find a particular solution $Y_i(t)$ for each subproblems. Then the sum $Y(t) = Y_1(t) + \cdots + Y_n(t)$ is a particular solution of the original nonhomogeneous equation (1.39).
- The general solution of Eq. (1.39) is

$$y = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) + Y(t). \tag{1.43}$$

for arbitrary constants $c_1, c_2, \ldots, c_n$ which can be determined with initial conditions.

1.8.3. An example

Check **Example 1** from **Lecture 8** and **Example 3** from **Lecture 6**.

1.9. nth order Linear nonhomogeneous equations: Method of Variation of Parameters

1.9.1. General Form

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t) \tag{1.44}$$

1.9.2. How to solve it?

- Find a fundamental set of solutions $y_1, y_2, \ldots, y_n$ of the corresponding homogeneous equation of (1.44).
- Find the Wronskian $W(t)$ of $y_1, \ldots, y_n$ and find W_m which is the determinant obtained from W by replacing the mth column by the column $(0, 0, \ldots, 0, 1)$.
- A particular solution of Eq. (1.44) is given by

$$Y(t) = \sum_{m=1}^{n} y_m(t) \int_{t_0}^{t} \frac{g(s)W_m(s)}{W(s)}\, ds, \tag{1.45}$$

where t_0 is a point on the interval where $p_1, p_2, \ldots, p_n$ are all continuous.

- The general solution of Eq. (1.44) is

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + Y(t). \tag{1.46}$$

for arbitrary constants $c_1, c_2, \ldots, c_n$ which can be determined with initial conditions.

1.9.3. An example

Check **Example 2** from **Lecture 8** and **Example 4** from **Lecture 6**.

1.10. Initial value linear problems with constant coefficients: Laplace transforms

1.10.1. General form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t) \tag{1.47}$$

where $a_0, a_1, \ldots, a_n$ are constants with initial conditions

$$y(0) = k_0, y'(0) = k_1, \ldots, y^{(n)}(0) = k_n. \tag{1.48}$$

1.10.2. How to solve it?

- Apply Laplace transform $\mathcal{L}$ on both sides. Since $\mathcal{L}$ is an operator, therefore the constants can be pulled outside.

$$a_0 \mathcal{L}\{y^{(n)}\} + a_1 \mathcal{L}\{y^{n-1}\} + \cdots + a_{n-1} \mathcal{L}\{y'\} + a_n \mathcal{L}\{y\} = \mathcal{L}\{g(t)\} \tag{1.49}$$

- Use the formula

$$\mathcal{L}\{y^{(n)}\} = s^n \mathcal{L}\{y\} - s^{n-1} y(0) - \cdots - s y^{(n-2)}(0) - y^{(n-1)}(0). \tag{1.50}$$

to write the left side of Eq. (1.49) in terms of $\mathcal{L}\{y\}$. Take any term that doesn't involve $\mathcal{L}\{y\}$ to the right side of the equation.
- The equation will look like

$$\mathcal{L}\{y\} G(s) = H(s) \tag{1.51}$$

where G and H are functions of s. Dividing both sides by $G(s)$ we have

$$\mathcal{L}\{y\} = \frac{H(s)}{G(s)} = F(s) \tag{1.52}$$

- Now

$$y = \mathcal{L}^{-1}\{F(s)\} \tag{1.53}$$

- $F(s)$ can be expressed as a sum of several terms (usually using partial fractions),

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s). \tag{1.54}$$

each of which are simpler and easier to find the inverse Laplace transform of using the table in the text book.

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F_1(s)\} + \mathcal{L}^{-1}\{F_2(s)\} + \cdots + \mathcal{L}^{-1}\{F_n(s)\}. \tag{1.55}$$

1.10.3. An example

Check **Example 4** from **Lecture 9** and **Lecture 10**.

1.11. Second order Linear Equations: Series Solutions

1.11.1. General Form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0. \tag{1.56}$$

1.11.2. How to solve it?

- We solve Eq. (1.56) in the neighborhood of an ordinary point x_0 ($P(x_0) \neq 0$), and when P, Q and R are polynomials of x.
- Assume that the solution has the form

$$y = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n. \tag{1.57}$$

Find

$$y' = a_1 + 2a_2(x - x_0) + \cdots + na_n(x - x_0)^{n-1} + \cdots = \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1}. \tag{1.58}$$

and

$$y'' = 2a_2 + \cdots + n(n-1)a_n(x - x_0)^{n-2} + \cdots = \sum_{n=2}^{\infty} n(n-1)a_n(x - x_0)^{n-2}. \tag{1.59}$$

- Substituting them back in Eq. (1.56) we get

$$P(x)\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} + Q(x)\sum_{n=1}^{\infty} na_n(x-x_0)^{n-1} + R(x)\sum_{n=0}^{\infty} a_n(x-x_0)^n = 0. \tag{1.60}$$

- By equating the coefficients of x^n we can find a recurrence relation involving the coefficients a_n. Solve this recurrence relation to find a_n.

1.11.3. An example

Check **Example 3** from **Lecture 12**.

1.12. Second order Linear Equations: Euler Equations

1.12.1. General Form

$$x^2y'' + \alpha xy' + \beta y = 0, \tag{1.61}$$

where α and β are real constants.

1.12.2. How to solve it?

- Form the characteristic equation

$$r^2 + (\alpha - 1)r + \beta = 0. \tag{1.62}$$

- Solve Eq. (1.23) to find the two roots r_1 and r_2.
  - If r_1 and r_2 are real and $r_1 \neq r_2$, the general solution is

$$y = c_1|x|^{r_1} + c_2|x|^{r_2}. \tag{1.63}$$

  - If r_1 and r_2 are real and $r_1 = r_2$, the general solution is

$$y = c_1|x|^{r_1} + c_2|x|^{r_1}\ln|x|. \tag{1.64}$$

  - If r_1 and r_2 are complex then let $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, the general solution is

$$y = c_1|x|^{\lambda}\cos(\mu\ln|x|) + c_2|x|^{\lambda}\sin(\mu\ln|x|). \tag{1.65}$$

    for arbitrary constants c_1 and c_2 which can be determined with initial conditions.

1.12.3. An example

Check **Example 2** from **Lecture 13**.