Notes on Calculus

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1 Introduction

Definition 1.1. The absolute value of a number x, written $|x|$ is the distance from 0 to x on the real line. Distances are always positive, so $|x| \geq 0$ for all $x \in \mathbb{R}$. The function $f(x) = |x|$ is defined as

$$f(x) = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$$

Definition 1.2. An even function is a function $f : A \to B$ such that $f(-x) = f(x)$ for every $x \in A$. An odd function is a function $f : A \to B$ such that $f(-x) = -f(x)$ for every $x \in A$.

Definition 1.3. A function f is increasing on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I. A function f is decreasing on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.

1.1 Algebra of functions

Definition 1.4. Let $f : A \to Y$, $g : B \to W$ be functions. Then we define $(f + g)(x) = f(x) + g(x)$ where the domain is $A \cap B$. We define $(f - g)(x) = f(x) - g(x)$ where the domain is $A \cap B$. We define $(fg)(x) = f(x)g(x)$, where the domain is $A \cap B$. We define $(f/g)(x) = f(x)/g(x)$, where the domain is $\{x \in A \cap B : g(x) \neq 0\}$.

Definition 1.5. Let $f : A \to B$, $g : X \to Y$. We define the composite function $f \circ g$ as $(f \circ g)(x) = f(g(x))$. The domain of the composition is all $x \in X$ in which $g(x)$ is defined which goes to an $a \in A$ in which $f(a)$ is defined. So, the domain is the intersection of a subset of Y, the range of $g(x)$, and the set A; that is, the domain of $f \circ g$ is $Range(g) \cap A$.

1.2 Limits

Definition 1.6. We say that f approaches L as x approaches a if and only if for each $e > 0$ given, there is a $d > 0$ such that

$$0 < |x - a| < d \implies |f - L| < e.$$

Definition 1.7. We say that f approaches L from the left as x approaches a from the left if and only if for each $e > 0$ given, there is a $d > 0$ such that

$$a - d < x < a \implies |f - L| < e.$$

Definition 1.8. We say that f approaches L from the right as x approaches a from the right if and only if for each $e > 0$ given, there is a $d > 0$ such that

$$a < x < a + d \implies |f - L| < e.$$

Something approaches something else if and only if the distance in between them disappears "as time goes by." So in the definitions above, notice that if the distance between x and a becomes small, then so must the distance between f and L.

Definition 1.9. The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true: $\lim f = \infty$ as $x \to a$, $\lim f = \infty$ as $x \to a^-$, $\lim f = \infty$ as $x \to a^+$, $\lim f = -\infty$ as $x \to a$, $\lim f = -\infty$ as $x \to a^-$, $\lim f = \infty$ as $x \to a^+$.

Theorem 1.1. Let c be a constant and suppose $\lim f$ and $\lim g$ as $x \to a$ exist. Then the following are true: (1) $\lim(f + g) = \lim f + \lim g$; (2) $\lim(f - g) = \lim f - \lim g$; (3) $\lim(cf) = c \lim f$; (4) $\lim(fg) = \lim f \lim g$; (5) $\lim f/g = (\lim f)/(\lim g)$ as long as $\lim g \neq 0$; (6) $\lim f^n = (\lim f)^n$, where $n > 0$; (7) $\lim c = c$; (8) $\lim x = a$ (as $x \to a$); (9) $\lim x^n = a^n$, as $x \to a$ and $n > 0$; (10) $\lim \sqrt[n]{x} = \sqrt[n]{a}$ as $x \to a$ and $n > 0$, and once $n > 0$, assume $a > 0$. (11) $\lim \sqrt[n]{f}$ $= \sqrt[n]{\lim f}$ as $x \to a$ and $n > 0$, and once n is even, assume $f > 0$.

Exercise 1.1. If p is a polynomial, show $\lim p(x) = p(m)$ as $x \to m$. *Proof.* The general formula for a polynomial is

$$p(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n.$$

We now apply the limit laws

$$\begin{aligned}\lim p(x) &= \lim a_0 + \lim a_1 x + \lim a_2 x^2 + ... + \lim a_n x^n \\ &= a_0 + a_1 m + a_2 m^2 + ... + a_n m^n \\ &= p(m)\end{aligned}$$

This shows that every polynomial is continuous, but we define continuity a little later. Before, we do exercises to practice our understanding of limits.

Exercise 1.2. If r is a rational function, show that $\lim r(x) = r(m)$, as $x \to m$, for every number m in the domain of r. *Proof.* From the previous exercise, we have that every polynomial is continuous. A rational function is the quotient of two polynomials; so, the problem is very similar. We can represent the limit of a rational function as

$$\begin{aligned}
\lim r(x) &= \lim \left(\frac{a_0 + a_1 x + a_2 x^2 + ... + a_n x^n}{b_0 + b_1 x + b_2 x^2 + ... + b_n x^n} \right) \\
&= \frac{\lim a_0 + \lim a_1 x + \lim a_2 x^2 + ... + \lim a_n x^n}{\lim b_0 + \lim b_1 x + \lim b_2 x^2 + ... + \lim b_n x^n} \\
&= \frac{a_0 + a_1 m + a_2 m^2 + ... + a_n m^n}{b_0 + b_1 m + b_2 m^2 + ... + b_n m^n} \\
&= r(m)
\end{aligned}$$

Theorem 1.2. A function $f \to L$ as $x \to a$ if and only if $f \to L$ as $x \to a^-$ and $f \to L$ as $x \to a^+$.

Theorem 1.3. If $f \leq g$ when x is near a, but $x \neq a$, and the $\lim f$ and $\lim g$ both exist as $x \to a$, then $\lim_{x \to a} f \leq \lim_{x \to a} g$.

1.3 Continuity

Definition 1.10. A function is continuous if and only if $\lim\ f(x) = f(a)$ as $x \to a$.

Definition 1.11. A function is continuous from the right at a number a if and only if $\lim\ f(x) = f(a)$ as $x \to a$ from the right. A function is continuous from the left at a if and only if $\lim\ f(x) = f(a)$ as $x \to a$ from the left.

Theorem 1.4. Let f be continuous at b. Let $\lim\ g(x) = b$ as $x \to a$. Then $\lim\ f(g(x)) = f(\lim g(x))$ as $x \to a$.

Theorem 1.5. Let g be continuous at a. Let f be continuous at $g(a)$. Then $f \circ g$ is continuous at a. That is, $f(g(x))$ is continuous at a.

Theorem 1.6 (intermediate value theorem)**.** Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$. Then there there exists a number c in $[a, b]$ such that $f(c) = N$.

Exercise 1.3. Prove that f is continuous at a if and only if $\lim\ f(a + h) = f(a)$ as $h \to 0$.

Exercise 1.4. To prove that sine is continuous, we need to show that lim $\sin(x) = \sin(a)$ as $x \to a$ for every real number a. We know that a function is continuous at a if and only if lim $f(a+h) = f(a)$ as $h \to 0$. Using the facts that $\lim \cos(x) = 1$ as $x \to 0$ and lim $\sin(x) = 0$ as $x \to 0$, show that $\sin(x)$ is continuous.

Exercise 1.5. Prove that cosine is a continuous function.

Exercise 1.6. For what values of x is f continuous?

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

Exercise 1.7. Is there a number that is exactly one more than its cube?

Exercise 1.8. Show that $|x|$ is continuous everywhere. Then prove that if f is continuous on an interval, then so is $|f|$. Now, if $|f|$ is continuous on an interval, is it true that f also is? If not, find a counterexample.

2 Derivatives

Definition 2.1. The derivative of a function f at a point $x = a$, written $f'(a)$ is defined to be

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

This definition comes from a similar definition. The derivative is the function that describes the slope of a tangent to a curve in some point. If we consider a curve f at some point a and you draw a tangent to f at the point a, then you can estimate the slope by

$$\frac{f(x) - f(a)}{x - a},$$

for some $x > a$; but that's an estimative. We would get a closer estimative if we choose a number x closer and closer to a; if we choose x exactly equal to a, we get a division by zero, so we can't do that. But we can consider the limit of the expression as x approaches a. If we do that, then we would be considering

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

which is the derivative of the function f at a point a, but where does the h come from in the definition above?

Let $h = x - a$, then $x = h + a$. Notice that as x approaches a, $x - a$ approaches 0. So if x approaches a, then h approaches 0 because $h = x - a$. So replace x by $h + a$ noticing that h approaches 0 as x approaches a, and write

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

which is exactly the first definition.

2.1 The chain rule

Suppose I write

$$f(x) = x^2.$$

This means that f is a function of x. Now, if I say: differentiate f with respect to t, someone would write

$$\frac{df}{dt} = 2x\frac{dt}{dx}$$

because

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}.$$

I'd like to justify this. Looking at Stewart, he says that it's reasonable to consider derivatives as rates of change; that's alright with me.

Also, he says that the derivative of a composition of functions $h(g(x))$ is the product of the derivatives h and g. So

$$[h(g(x)]' = h'(g(x))g'(x)x'$$

but since we're differentiating with respect to x, we get

$$[h(g(x)]' = h'(g(x))g'(x)$$

because $x' = 1$.

In the function f, however, where's t? I think that we have to suppose that x is a function of t: $x(t)$. So we rewrite f as

$$f(x) = x(t)^2.$$

Now, differentiating it, we need the chain rule which would say

$$\frac{df}{dt} = 2x(t)\frac{dx}{dt}.$$

But, the truth is that here x has no relationship with t — we didn't specify any, so it doesn't make any sense to draw such a relationship here. However, these relationships exist in related rates problems, so there to express the chain rule using Leibniz notation is useful and that's why we do it.

Now, I'd like to investigate why the chain rule is as it is, without resorting to a formal proof of it. I want an intuitive idea of why the rate of change of a composition is the product of the rate of change of each of the functions involved in the composition.

It's funny that the for a function $a(x) = b(x)c(x)$, the derivative of $a(x)$ is not the product of the derivatives of $b(x)$ and $c(x)$, but if a(x) = b(c(x)), then it is — the derivative of $a(x)$ is the product of the derivatives of $b(x)$ and $c(x)$.

Consider $f(x(t))$; that is, the composition of f and x and t. I want the rate of change of f with respect to t. But f is related to t through $x(t)$. So I reason, as Stewart: if x changes twice as fast compared to the changes in t, and f changes three times as fast compared to the changes in $x(t)$, then we would expect f to change six times as fast compared to t.

If t causes a double change in x, and x causes a triple change in f, then t causes a "sixple" change in f. This is a principle of mathematics. I think that this is called the Counting Principle.

The chain rule expresses a chain of dependences between functions. Such dependence is resolved by the operation of multiplication because the dependence is an instance of the counting principle.

The same does not happen to the product of two functions. The derivative of $f(x)g(x)$ cannot be obtained by an application of the chain rule. If so, then in order for us to be able to differentiate any function, we need the product rule, and the chain rule.

2.2 Related rates

Related rates involve direct applications of the chain rule. Suppose that oxygen is being pumped into a balloon — let it be spherical, so that we can use the volume of a sphere which is widely known. Suppose that the rate which oxygen goes in is 10 cubic meters per second — whatever. Now, as the balloon grows in volume, the volume grows — evidently — and the radius of the balloon also grows. We may ask: how fast is the radius growing when the diameter of the balloon is 1 meter?

The rate at which oxygen is pumped is a derivative

$$\frac{dV}{dt} = 10\frac{m^3}{s}.$$

We know that volume of a sphere is described by

$$V = \frac{4}{3}\pi r^3$$

which connects volume to the radius. We want the rate of growth of the radius, so we're looking for a derivative of the volume with respect to the radius; we were given the derivative of the volume with respect to time. We then express the change in volume with respect to the change in the radius by using the chain rule. That is,

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dr}\frac{dr}{dt} \\ &= 4\pi r^2\frac{dr}{dt} \end{aligned}$$

Connection established. We now solve for dr/dt.

$$\frac{dV}{dt}\frac{1}{4\pi r^2} = \frac{dr}{dt}.$$

But since $\frac{dV}{dt} = 10\frac{m^3}{s}$, we have

$$\frac{10m^3/s}{4\pi r^2} = \frac{dr}{dt}.$$

When the diameter is 1 meter, the radius is 0.5 meters, so

$$\frac{10\frac{m^3}{s}}{4\pi(0.5)^2} = \frac{dr}{dt}$$
$$\frac{10}{\pi} = \frac{dr}{dt},$$

so the radius of the balloon is growing at a speed of $\frac{10}{\pi}$ meters per second.

2.3 Linearizations

Suppose that we want to find the tangents to

$$\frac{x}{x+1}$$

that touch the point (1,2). We would first compute the derivative of the curve which is

$$\frac{1}{(x+1)^2}.$$

Now, these tangents will look like

$$y - f(a) = f'(a)(x-a).$$

This is the point-slope formula of a line. We know that the slope of a line is found by using two points of the line and computing

$$\frac{y_2 - y_1}{x_2 - x_1} = s,$$

so $y_2 - y_1 = s(x_2 - x_1)$. Here, however, we only have one point, but we also have a form for the slope. At a point $x = a$, we have

$$y_2 - f(a) = \frac{1}{(a+1)^2}(x_2 - a),$$

so for these tangents to touch (1,2) it must be true that

$$2 - \frac{a}{(a+1)} = \frac{1}{(a+1)^2}(1-a).$$

Now, which numbers a are solutions to this equation? They're $-2-\sqrt{3}$, and $-2+\sqrt{3}$. This gives us two lines; they are

$$y - \frac{(-2-\sqrt{3})}{(-2-\sqrt{3}+1)} = \frac{(x-(-2-\sqrt{3}))}{(-2-\sqrt{3}+1)}^2$$
$$y - \frac{(-2+\sqrt{3})}{(-2+\sqrt{3}+1)} = \frac{(x-(-2+\sqrt{3}))}{(-2+\sqrt{3}+1)}^2.$$

Now, because these two lines are tangents to the curve, they share nearby points when x is close enough to 1. So, under these circumstances, we may say that

$$f(x) = L(x) = f(a) + f'(a)(x-a),$$

for x nearby $x = a$.

2.4 Differentials

If $y = f(x)$, where f is a differentiable function, then the differential dx is an independent variable; that is, dx can be given the value of any real number. The differential dy is then defined in terms of dx by the equation

$$dy = f'(x)dx$$

So dy is a dependent variable; it depends on the values of x and dx. If dx is given a specific value and x is taken to be some specific number in the domain of f, then the numerical value of dy is determined.

Suppose that P is a point on a curve $f(x)$ with coordinates $(x, f(x))$ and that Q is some point a little further on the curve with coordinates $(x+\Delta x, f(x+\Delta x))$. Let $dx = \Delta x$. Then, the change in y is

$$\Delta y = f(x + \Delta x) - f(x).$$

Certainly, because $f(x + \Delta x)$ is the height of the point Q; then, if we subtract the height of the point P, we get the change in y as we moved from P to Q.

Now, dy expresses the change in the linearization of $f(x)$, which for a concave down curve with positive slope, would be larger than Δy.

In a concave down graph with positive slope, for example, let $L(x)$ be a tangent to the curve at P. So $\Delta y < dy$. The value of Δy is simply $f(x+\Delta x) - f(x)$, so the value of dy should be something like $L(x + \Delta x) - L(x)$. However, the value of dy is $f'(x)dx$. So it must follow that

$$L(x + \Delta x) - L(x) = f'(x)\ dx.$$

2.5 More on differentiation

Theorem 2.1 (extreme value theorem)**.** If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Theorem 2.2 (fermat's theorem)**.** If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Proof. We need to assume a local maximum and show the conclusion, then a local minimum and show the conclusion, but both phases are similar, so we'll write only one.

Let $f(c)$ be a local minimum. So $f(c) \leq f(x)$. Later. □

Definition 2.2. A critical number of a function f is a number c in the domain of f such that $f'(c)$ or $f'(c)$ does not exist.

Theorem 2.3 (fermat's theorem)**.** If has a local maximum of minimum at c, then c is a critical number of f.

From this, and supposing that we have implemented an algorithm to compute derivatives of functions, then we can write a program that can answer the critical numbers of a function.

Theorem 2.4. Let $f(x)$ be continuous on a closed interval $[a, b]$. (1) Compute all critical points in (a, b). (2) Compute the values of f at

$$x = a \text{ and } x = b.$$

Now, the largest computed value in (2) and (3) is the absolute maximum; the smallest is the absolute minimum.

Exercise 2.1. Show that 5 is a critical number of

$$g(x) = 2 + (x - 5)^3,$$

and show that $g(x)$ has no local extremum at 5.

Solution. A critical number c is one in which $g'(c) = 0$ or $g'(c)$ does not exist. We have $g'(5) = 0$, so 5 is a critical number.

Assume that 5 is a local maximum. Then $g(5) \geq g(x)$ for all x in some open interval $(5 - d, 5 + d)$, where $d > 0$. Let $x < 5$. Then

$$\begin{aligned} x - 5 &< 0 \\ (x - 5)^3 &< 0 \\ 2 + (x - 5)^3 &< 2. \end{aligned}$$

Let $x > 5$. Then

$$\begin{aligned} x - 5 &> 0 \\ (x - 5)^3 &> 0 \\ 2 + (x - 5)^3 &> 2. \end{aligned}$$

So there's no neighborhood of 5 such that $g(5) \geq 2$.

Exercise 2.2. Prove that $x^3 + x - 1 = 0$ has exactly one real root.

Solution. Let $f(x) = x^3 + x - 1$. Notice that $f(0) = -1$ and that $f(1) = 1$. Since polynomials are continuous, then there is a c in $(0, 1)$ such that $f(c) = 0$, so c is a root.

Now, suppose that there are two roots, a and b. Since they're roots, it's true that $f(a) = 0 = f(b)$. Notice that f is continuous on $[a, b]$ and differentiable at (a, b) so there's a number d in $[a, b]$ such that $f'(d) = 0$. But $f'(x) = 3x^2 + 1 \geq 0$ for all x, so no real d is possible, and therefore, it's false that there are two roots, so there's only one.

Exercise 2.3. If $f(x)$ has a minimum value at c, show that $-f(x)$ has a maximum value at c.

Solution. If f has a minimum at c, then $f(c) \leq f(x)$ for all x in the domain of f. Multiply this inequality by -1 to get $-f(c) \geq -f(x)$ for all x in the domain of f. So c is a maximum for $-f(x)$.

Exercise 2.4. A cubic function is a polynomial of degree 3; that is, it has the form $f(x) = mx^3 + nx^2 + ox + p$, where $m \neq 0$. Show that a cubic function can have no more than two critical numbers. How many local extrema can a cubic function have?

Solution. Polynomials are differentiable, so there is no real number c to which $f'(c)$ does not exist. Notice that $f'(x) = 3mx^2 + 2nx + o$ which is is a quadratic equation which has only two roots maximum, so no more than two critical numbers is possible. Every local extremum is a critical number, so no more than two would be possible.

2.6 The mean value theorem

Rolle's theorem says that a differentiable function that goes up and down has some c such that $f'(c) = 0$. The up and down criteria is that $f(a) = f(b)$ for a domain $[a, b]$. So let's write the theorem.

Theorem 2.5. Let f be a function that satisfies: f is continuous on the closed interval $[a, b], f$ is differentiable on the open interval (a, b), $f(a) = f(b)$. Then there is a number c in (a, b) such that $f'(c) = 0$.

Proof. We prove in three cases. Case one: let $f(x)$ be a constant k. Then $f'(x) = 0$ and so any number c in (a, b) can be taken. Case two: let $f(x) > f(a)$ for some x in (a, b). I'm bored here. Let's move on. □

The Mean Value Theorem should be called perhaps the Same Slope Theorem. I don't really know why the word "mean" was chosen; perhaps there's more to it than what I know right now. The theorem says that if f is very nice on $[a, b]$, then there is a number in the interval to which a tangent to the curve has the same slope as the secant AB — that is, a secant line from the a to b. The tangent would be parallel to this secant line, then.

Theorem 2.6 (mean value theorem)**.** Let f be continuous on $[a, b]$ and differentiable on (a, b). There is a c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Exercise 2.5. Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all x. How large can $f(2)$ possibly be?

Solution. We're given one point and a restriction for the slope of the function. Then we're asked how large can $f(2)$ be? Geometrically, plot the number $(0, -3)$ and take your ruler and position it as if you were going to draw a line with slope 5 — the maximum slope you can. Then, see how large $f(2)$ could be.

Now, precisely. In the interval $[0, 2]$, there is a number c such that

$$\begin{aligned} f'(c) &= \frac{f(2) - f(0)}{2 - 0} \\ &= \frac{f(2) + 3}{2}. \end{aligned}$$

But we're looking for the highest value of $f(2)$, so we will use our highest slope possible: 5. That is,

$$\begin{aligned} 5 &= \frac{f(2)+3}{2} \\ 10 &= f(2)+3 \\ 7 &= f(2). \end{aligned}$$

Très mignon, non?

Theorem 2.7. Let $f'(x) = 0$ for all x in (a, b). Then f is constant.

Proof. Let x_1 and x_2 be any two distinct numbers in (a, b), so $x_1 < x_2$. Since f is differentiable on (a, b) it is differentiable on (x_1, x_2) and therefore continuous on $[x_1, x_2]$. By the mean value theorem, there's c in $[x_1, x_2]$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(x) = 0$ for all x, then $f'(c) = 0$. So

$$\begin{aligned} 0 &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ &= f(x_2) - f(x_1) \\ f(x_1) &= f(x_2). \end{aligned}$$

So f has the same value at any two distinct numbers x_1, x_2 in (a, b), which by definition means that f is constant. □

Corollary 2.8. Let $f'(x) = g'(x)$ for all x in (a, b). Then $f - g$ is constant on (a, b). That is, $f(x) = g(x) + c$ where c is a constant.

Proof. Let $F(x) = f(x) - g(x)$. Then

$$F'(x) = f'(x) - g'(x) = 0$$

for all x in (a, b). So $f(x) - g(x)$ is constant. □

Exercise 2.6. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$.

Exercise 2.7. Let $f(x) = \sqrt[3]{x}$ be defined on $[0, 1]$. Find all values that satisfy the mean value theorem.

Solution. The function is differentiable on $(0, 1)$, so by the mean value theorem there is a c in $(0, 1)$ such that

$$\begin{aligned} f'(c) &= \frac{f(0) - f(1)}{-1} \\ f'(c) &= 1. \end{aligned}$$

Which number c would be this one? It's true that $f'(x) = \frac{x^{-2/3}}{3}$. So

$$\begin{aligned}\frac{c^{-2/3}}{3} &= 1\\ c^{-2/3} &= 3\\ \frac{1}{\sqrt[3]{c^2}} &= 3\\ \frac{1}{c^2} &= 27\\ \frac{1}{27} &= c^2\\ \pm\sqrt{\frac{1}{27}} &= c.\end{aligned}$$

But only $\sqrt{\frac{1}{27}}$ is applicable because $0 < c < 1$.

Exercise 2.8. Show that $x^5 + 10x + 3 = 0$ has exactly one real root.

Solution. Let $p(x) = x^5 + 10x + 3$. We first show that there is a real root. Notice that $p(-1) = -8$ and $p(1) = 14$. By the intermediate value theorem, there is a c in $(-1, 14)$ such that $p(c) = 0$, so c is a root.

Now suppose that there are two roots: a and b. Since they're roots, it's true that $f(a) = 0 = f(b)$. So, by Rolle's theorem, there is a d in (a, b) such that $f'(d) = 0$. Let's find d. Since $p'(x) = 5x^4 + 10 \geq 0$ for all x, it turns out that no such d is possible. Therefore, there is no two real roots a and b. So there's only one.

Exercise 2.9. Show that $x^5 - 6x + c = 0$ has only one root in $[-1, 1]$.

Solution. Let $f(x) = x^5 - 6x + c$. Suppose $f(x) = 0$ has two roots a and b in $[-1, 1]$. Then $f(a) = 0 = f(b)$ so by Rolle's theorem there is a d in (a, b) with $f'(d) = 0$. So $0 = f'(d) = 5d^4 - 6$ which implies

$$d = \pm\sqrt[4]{\frac{6}{5}}$$

which are both outside (a, b) because (a, b) subset $[-1, 1]$ and d is not in $[-1, 1]$. So $f(x)$ can have at most one root in $[-1, 1]$.

This problem is different from the previous one because in the previous one, we're supposed to show that there is one root, and here there could be none in that particular interval. For example, if $c = 5$, we have $f(x) = x^5 - 6x + 5$, so $x = 1$ is a root, but if $c > 5$, then there are no more roots in that interval. Similarly, if $c = -5$, we have $f(x) = x^5 - 6x - 5$ and so $x = -1$ is a root, but if $c < -5$, then there are no more roots in that interval.

A good exercise here is to ponder how someone would come up with a question like that. That is, how to construct equations from roots in a particular interval such that the equation has a particular degree?

Take a function such as $f(x) = x^2 - 1$. We know that the roots of this function are -1 and 1, so

$$F(x) = \int f(x)\ dx = \frac{x^3}{3} - x + c$$

will have local extrema at -1 and 1. Now, if $F(-1) > 0$ and $F(1) < 0$, then by the intermediate value theorem, we know that there is a number $-1 < d < 1$ such that $F(d) = 0$. Because we constructed $F(x)$ from $f(x)$, we know that $F(x)$ has only the local extrema we want, so no more roots can exist in that interval.

Now, how can we make sure that $F(-1) > 0$ and $F(1) < 0$? Well, at -1, we have

$$\frac{-1^3}{3} + 1 + c > 0,$$

so

$$c > \frac{1}{3} - 1 = \frac{-2}{3},$$

and at 1, we have

$$\frac{1^3}{3} - 1 + c < 0,$$

so

$$c < -\frac{1}{3} + 1 = \frac{2}{3}.$$

So we need a number c such that

$$-\frac{2}{3} < c < \frac{2}{3}.$$

Now, suppose that you want your final equation to have a particular degree n. Then, look for a derivative of degree $n - 1$. For example, suppose that $n = 101$. Then we need a function of degree 100. For example, $x^{100} - 1$ will have roots at -1 and 1. What if you want to work with an arbitrary interval for the root to fall in?

Suppose you want an interval like $(-m, m)$. Then let $x = m$ and

$$\begin{aligned} x &= m \\ x - m &= 0 \\ (x - m)^2 &= 0. \end{aligned}$$

So evidently, we have $\pm m$ as a root. Then we integrate the function to get

$$F(x) = \frac{x^3}{3} - mx^2 + xm^2 + c.$$

Now by the intermediate value theorem, this is a function which has a root in $(-m, m)$ for some particular values of c. To find out which values of c we need to choose, we compute

$$\begin{aligned} \frac{-m^3}{3} - m(-m^2) + (-m)m^2 + c &< 0 \\ \frac{-m^3}{3} - 2m^3 + c &< 0 \\ c &< \frac{m^3}{3} + 2m^3. \end{aligned}$$

For the other part,

$$\begin{aligned}\frac{m^3}{3} - m(m^2) + (m)m^2 + c &> 0\\ \frac{m^3}{3} - m^3 + m^3 + c &> 0\\ c &> \frac{-m^3}{3}.\end{aligned}$$

So,

$$\frac{-m^3}{3} < c < \frac{m^3}{3} + 2m^3.$$

I wouldn't believe this if I were you. This seems rather fallacious.

Exercise 2.10. Show that $x^4 + 4x + c = 0$ has at most two real roots.

Solution. A similar reasoning. Assume it has three: a, b, c. Then it's true that $0 = f(a) = f(b) = f(c)$. So by Rolle's theorem, there is a d in (a, b) such that $f'(d) = 0$, and there is an e in (b, c) such that $f'(e) = 0$. The derivative of the function is $g(x) = x^3 + 4$, and we must find two reals roots of this function; they would be d and e. However, one root of $g(x)$ is

$$x = \sqrt[3]{-4},$$

because

$$\begin{aligned}x^3 + 4 &= 0\\ x^3 &= -4\\ x &= \sqrt[3]{-4}.\end{aligned}$$

Now, let $x < \sqrt[3]{-4}$. Then

$$\begin{aligned}x^3 &< -4\\ x^3 + 4 &< 0,\end{aligned}$$

and let $x > \sqrt[3]{-4}$, so

$$\begin{aligned}x^3 &> -4\\ x^3 + 4 &> 0.\end{aligned}$$

So to the left of $\sqrt[3]{-4}$, $g(x)$ is negative, and to the right $g(x)$ is positive. So, $g(x)$ will touch the x-axis only once; there can't be two real roots, then. Therefore, $f(x)$ has at most two real roots.

Exercise 2.11. Show that a polynomial of degree 3 has at most 3 real roots.

Solution. A polynomial of degree 3 is of the form

$$ax^3 + bx^2 + cx + d.$$

Assume that it has four real roots. Then it's true that

$$0 = f(a) = f(b) = f(c) = f(d).$$

So by Rolle's theorem, there are numbers p in $(a, b), q$ in (b, c), r in (c, d) such that $0 = f'(p) = f'(q) = f'(r)$. But $f'(x)$ is a second degree polynomial with three real roots which is not possible.

Exercise 2.12. Show that a polynomial of degree n has at most n real roots.

Solution. We know that a polynomial of degree 1 has 1 real root. So, if we try to prove this by induction, at least we know that the proposition is true for $n = 1$.

Now, suppose that the proposition is true for some $n = k$. Let $p(x)$ be of degree $k + 1$. Suppose that $p(x)$ has more than $k + 1$ real roots; they are $r_1, r_2, ..., r_{k+1}, r_{k+2}$. Then

$$0 = p(r_1) = ... = p(r_{k+2})$$

and by Rolle's theorem it's true that

$$0 = p'(c_1) = ... = p'(c_{k+1}).$$

But if p is of degree $k + 1$, then p' is of degree k, and cannot have $k + 1$ roots. So it's not true that $p(x)$ has more than $k + 1$ real roots. Therefore, it has less or equal to $k + 1$ roots.

Proofs by induction may seem fallacious sometimes. One could object here that we cannot base our argument on the fact that p' of degree k cannot have $k + 1$ roots because it is the same thing that we're trying to prove.

That's an unfair objection, however, because we tested for $n = 1$ and it was true; then we assumed to be true for all polynomials of kth degree that they don't have $k + 1$ roots, so it's true under that scope of the proof. From that assumption, we showed that the same could not happen to p of degree $k+1$ and with everything altogether, we concluded that a polynomial of degree n has at most n roots.

In fact, I seldom use the letter n twice in a proof by induction; instead, I use k for the assumption when I'm trying to show that something is true for $k + 1$. This way I avoid getting confused with the n of the proposition and raise the objection that I'm assuming what I'm trying to prove.

Exercise 2.13. Suppose that f is differentiable on $\mathbb{R}$ and has two roots. Show that f' has at least one root.

Solution. If this proposition is true, then it means that a differentiable function of degree 2 or more changes direction at at least in one point. Call the roots of f by a and b. So

$$f(a) = 0 = f(b).$$

By Rolle's theorem, we have that $f'(c) = 0$ for some c in (a, b).

Exercise 2.14. Suppose that f is twice differentiable on $\mathbb{R}$ and f has three roots. Show that f'' has at least one real root.

Solution. Let a, b, and c be in $\mathbb{C}$. So

$$0 = f(a) = f(b) = f(c).$$

By Rolle's theorem, we have $f'(d) = 0$ for some d in (a, b) and $f'(e) = 0$ for some e in (b, c). So these are two real roots of f'. Since f is twice differentiable, we can differentiate f' to get f''. Since $f'(d) = 0 = f'(e)$, then by Rolle's theorem, there is an i in (d, e) such that $f''(i) = 0$, so f'' has at least one real root. We may try to generalize this.

Theorem 2.9. Suppose that f has n roots, and is differentiable on $\mathbb{R}n - 1$ times. Then $f^{(k)}$ has $n - k$ real roots where $0 < k \leq n - 1$.

Let's test our generalization. Let $f(x) = x^7 - 1$. This polynomial has 7 roots and is $7 - 1$ times differentiable on $\mathbb{R}$. Then, by our generalization, it must be true that $f'(x) = x^6$ has $7-1$ real roots, $f''(x) = x^5$ must have $7-2$ real roots, $f^{(3)}(x) = x^4$ must have $7 - 3$ real roots and so on until $f^{(6)} = x$ which must have one real root. The generalization seems to work. Can we prove this now? Hmm, I don't know.

2.7 Even more on differentiation

It's interesting to notice that the mean value theorem is used in many applications. There is a variety of problems in which the mean value theorem can be used; one challenge is to recognize when you can use it; we'll describe some interesting problems in which the mean value theorem can be used as a path to the solution. Let's start by looking at the proof of the following theorem.

Theorem 2.10. If $f'(x) > 0$ on some interval B, then f is increasing on B. If $f'(x) < 0$ on B, then f is decreasing on B.

Proof. Let x_1, x_2 be in B such that $x_1 \neq x_2$, so we may assume that $x_1 < x_2$. We must show that $f(x_1) < f(x_2)$. Assume that $f'(x) > 0$, and notice that f is differentiable on $[x_1, x_2]$. So by the mean value theorem, there is a c in $[x_1, x_2]$ such that

$$f'(c)(x_2 - x_1) = f(x_2) - f(x_1).$$

Notice that $f'(c) > 0$ and notice that $x_2 - x_1 > 0$ too. So the left side of the equation above is positive and therefore the right side must be as well; that is, $f(x_2) > f(x_1)$, and we're done. For a decreasing function, we need only repeat the proof flipping the notation around. □

Let's see more problems in which I think that the mean value theorem would provide a way to a solution. The next three are of a pretty interesting type of problem, and we already solved one of the same kind, I think.

Exercise 2.15. Suppose $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ possibly be?

Exercise 2.16. Suppose f is continuous on $[2, 5]$ and $1 \leq f'(x) \leq 4$ for all x in $(2, 5)$. Show that $3 \leq f(5) - f(2) \leq 12$.

Exercise 2.17. Does there exist a function f such that $f(0) = -1, f(2) = 4$, and $f'(x) \leq 2$ for all x?

Exercise 2.18. Show that $\tan x > x$ for $0 < x < \pi/2$.

Solution. We're going to show that $\tan(x) - x > 0$. We know that

$$\tan(x) = \frac{\sin(x)}{\cos(x)}.$$

As $x \to 0, \tan(x) \to 0$. As $x \to \pi/2, \tan(x) - x \to \infty$ so if I can show that $\tan(x) - x$ is increasing on $(0, \pi/2)$, then we know that

$$f(x) = \tan(x) - x > 0$$

is true for all x in $(0, \pi/2)$ which makes $\tan(x) > x$ true for all x in $(0, \pi/2)$. We look into $f'(x)$ then. We have

$$f'(x) = \sec^2(x) - 1 = 1/\cos^2(x) - 1$$

which is only zero when $\cos^2(x) = 1$ which happens at $x = 0$, but 0 is not in $(0, \pi/2)$. So, for x in $(0, \pi/2)$, $f'(x)$ is always positive. Therefore, $\tan(x) - x > 0$ for all x in $(0, \pi/2)$ and so $\tan(x) > x$ for all x in $(0, \pi/2)$.

Exercise 2.19. Prove that for all $x > 1$,

$$2\sqrt{x} > 3 - \frac{1}{x}.$$

Solution. We will show that $2\sqrt{x} - 3 + 1/x > 0$ for $x > 1$. When $x = 1$ we have $f(x) = 0$. As $x \to \infty, f(x) \to \infty$. It's true that

$$f'(x) = \frac{1}{\sqrt{x}} - \frac{1}{x^2}.$$

Now, when $f'(x) = 0$? Never. We have that

$$\frac{1}{\sqrt{x}} > \frac{1}{x^2},$$

for all $x > 1$. So $f'(x) > 0$ for all $x > 1$. So $f(x)$ is always increasing on $(1, \infty)$, so $2\sqrt{x} > 3 - 1/x$ for all $x > 1$.

Exercise 2.20. Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b). Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$. Prove that $f(b) < g(b)$.

Solution. We're going to consider $h(x) = f(x) - g(x)$. What's $h(a)$? It's

$$h(a) = f(a) - g(a) = 0,$$

because $f(a) = g(a)$. It's true that $h(x)$ is continuous and differentiable on (a, b), so we can apply the mean value theorem to say that there exists a number c in (a, b) such that

$$\begin{aligned} h'(c) &= f'(c) - g'(c) \\ &= \frac{h(b) - h(a)}{b - a} \\ &= \frac{f(b) - g(b) - f(a) + g(a)}{b - a}. \end{aligned}$$

We know that $b - a > 0$. We also know that $f'(c) - g'(c) < 0$ because $f'(x) < g'(x)$ for $a < x < b$. So it must be true that

$$f(b) - g(b) - f(a) + g(a) < 0,$$

otherwise $f'(c) - g'(c)$ would be positive. So, we write

$$\begin{aligned} f(b) - g(b) - f(a) + g(a) &< 0 \\ f(b) - g(b) - (f(a) - g(a)) &< 0 \\ f(b) - g(b) - 0 &< 0 \\ f(b) &< g(b), \end{aligned}$$

as desired.

Exercise 2.21. Show that $\sqrt{1+x} < 1 + x/2$ if $x > 0$.

Solution. We'll look into $f(x) = \sqrt{1+x} - 1 - x/2 < 0$. At 0, we have $\sqrt{1} - 1 = 0$. As $x \to \infty$, we have $f(x) \to -\infty$. So we must show that $f'(x)$ is always decreasing.

We have

$$f'(x) = \frac{1}{2\sqrt{1+x}} - \frac{1}{2}.$$

Is it ever zero? It is when $\sqrt{1+x} = 1$ which only happens when $x = 0$, but we don't care about $x = 0$. We must show then

$$2\sqrt{1+x} > 2.$$

That is, we can show that $\sqrt{1+x} > 1$ for all $x > 0$. It indeed is because $\sqrt{1+x} > 1$ implies $1 + x > 1$ which is true whenever $x > 0$. So $f'(x) < 0$ for all $x > 0$. So we're done.

Exercise 2.22. Suppose f is an odd function and is differentiable everywhere. Prove that for every positive number b, there exists a number c in $(-b, b)$ such that

$$f'(c) = \frac{f(b)}{b}.$$

Solution. By the mean value theorem, there's a c in $(-b, b)$ such that

$$f'(c) = \frac{f(b) - f(-b)}{b + b}.$$

But since f is odd, then we may rewrite that as

$$\begin{aligned} f'(c) &= \frac{f(b) + f(b)}{b + b} \\ &= \frac{2f(b)}{2b} \\ &= \frac{f(b)}{b}, \end{aligned}$$

as desired.

Exercise 2.23. Prove that $|\sin a - \sin b| \le |a - b|$ for all a and b in $\mathbb{R}$.

Solution. By the mean value theorem, there's a c in (a, b) such that

$$\sin'(c) = \frac{\sin(b) - \sin(a)}{b - a}.$$

We know that $\sin'(c) = \cos(c)$ and that $|\cos(c)| \le 1$. Now, take the absolute value of the equation above to get

$$\begin{aligned} |\cos(c)| &= \left| \frac{\sin(b) - \sin(a)}{b - a} \right| \\ &= \frac{|\sin(b) - \sin(a)|}{|b - a|} \\ |b - a||\cos(c)| &= |\sin(b) - \sin(a)|. \end{aligned}$$

Since $|\cos(c)| \le 1$, then it's true that

$$|b - a| \ge |\sin(b) - \sin(a)|.$$

Exercise 2.24. If $f'(x) = m$ for all x, where m is a constant, show that $f(x) = mx + n$ for some constant n.

Exercise 2.25. A number m is called a fixed point of f if $f(m) = m$. Prove that if $f'(x) \ne 1$ for all real numbers x, then f has at most one fixed point.

Theorem 2.11. Suppose f'' is continuous near c. Then if $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c, and if $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Notice that the theorem does not mention any case when $f''(c) = 0$. That's because we cannot say much when this is the case. The same situation happens when $f''(c)$ does not exist. But the first derivative handles all these difficulties.

Exercise 2.26. Assume that all functions in this proposition are twice differentiable and that their second derivatives are never zero. If f and g are concave upward on B, show that $f + g$ is concave upward on B.

Exercise 2.27. Assume that all functions in this proposition are twice differentiable and that their second derivatives are never zero. If f is positive and concave upward on B, show that $[f(x)^2]$ is concave upward on B.

Exercise 2.28. Assume that all functions in this proposition are twice differentiable and that their second derivatives are never zero. If f and g are positive, increasing, concave upward functions on B, show that fg is concave upward on B.

Exercise 2.29. Assume that all functions in this proposition are twice differentiable and that their second derivatives are never zero. Suppose that f and g are both concave upward on $(-\infty, \infty)$. Under what condition on f will the composite function $h(x) = f(g(x))$ be concave upward?

Exercise 2.30. Show that a cubic function always has exactly one point of inflection. If it has three roots, show that the x coordinate of the inflection point is $(x_1 + x_2 + x_3)\ /\ 3$.

Exercise 2.31. Prove that if $(m, f(m))$ is a point of inflection of f, and f'' exists in an open interval containing m, then $f''(m) = 0$. Stewart suggests: apply the first derivative test and fermat's theorem to the function $g = f'$.

Exercise 2.32. Show that if $f(x) = x^4$, then $f''(0) = 0$, but $(0, 0)$ is not an inflection point of the graph of f.

Exercise 2.33. Show that $g(x) = x|x|$ has an inflection point at $(0, 0)$ but $g''(0)$ does not exist.

Exercise 2.34. Suppose that f'' is continuous and $f'(m) = f''(m) = 0$, but $f''(m) > 0$. Does f have a local maximum or minimum at m? Does f have a point of inflection at m?

Exercise 2.35. Find a function f whose f' and f'' are always negative.

Exercise 2.36. Find a cubic function $f(x) = mx^3 + nx^2 + px + q$ that has a local maximum of 3 at -2 and a local mininum of 0 at 1.

Solution. The requirements for $f'(x)$ are $f'(-2) = 0, f'(1) = 0$ and $f'(x) = 3mx^2 + 2nx + p$. Also, $f(-2) = 3$ and $f(1) = 0$. So we need

$$\begin{aligned} -8m + 4n - 2p + q &= 3 \\ m + n + p + q &= 0 \\ 12m - 4n + p + 0 &= 0 \\ 3m + 2n + p + 0 &= 0. \end{aligned}$$

This system gives us $m = 2/9, n = 1/3, p = -4/3, q = 7/9$. So the function that we want is $f(x) = (2/9)x^3 + (1/3)x^2 - (4/3)x + (7/9)$.

2.8 Limits at infinity

Definition 2.3. Let f be a function defined on some interval (m, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that for every $e > 0$, there is a number n such that

$$\text{if } x > n \text{ then } |f(x) - L| < e.$$

Definition 2.4. Let f be a function defined on some interval $(-\infty, m)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that for every $e > 0$, there is a number n such that

$$\text{if } x < n \text{ then } |f(x) - L| < e.$$

Definition 2.5. Let f be a function defined on some interval (m, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

means that for every positive M, there is a number n such that

$$\text{if } x > n \text{ then } f(x) > M.$$

Theorem 2.12. If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0.$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0.$$

2.9 Drawing functions

We have enough tools to draw functions now. Let's look into the things which we need to draw functions. We need the critical numbers. With the critical numbers, we break the real line up into intervals delimited by the critical numbers. We then look at the first derivative of the function at each of these intervals; we then determine whether f is increasing on each of these intervals.

So we now have a clue of the way that the function travels in the plane of Descartes, but we need to know when it goes up and when it comes down and when it simply slows down and keeps going: we the extreme values; we need the local maxima, local minima and the absolute ones if they exist.

But to find the local extrema we need only the first derivative. We observe the critical points in which the function changes sign; that is, when it changes from increasing to decreasing or from decreasing to increasing.

2.9.1 An algorithm

Let $f(x)$ be a function. Determine the domain of $f(x)$. Then look for y-intercepts, if possible, and find out where the curve touches the y-axis. Mark these points.

Look for symmetry. Is it odd? Is it even? Find out, if you can, and keep that in mind making wise decisions: if it's even or odd, we draw only for $x \geq 0$ and we use a mirror later.

Look for periodicity. Is there a period? Which one?

Look for asymptotes. Are there vertical ones? Are there horizontal ones? Find them. Are there slant asymptotes? If so, find them.

Compute $f'(x)$. Find all critical numbers. Solve $f'(x) = 0$. Break the real line into as many intervals as possible from the solutions. Determine where $f'(x)$ is positive and where $f'(x)$ is negative. Find all local maxima and all local minima.

Compute $f''(x)$. Find all inflection points by computing whether $f''(x) > 0$ or $f''(x) < 0$. Look at $f'(x)$ when $f''(x) = 0$. Convince yourself that you need not a new division of the real line to look into $f''(x)$.

Draw $f(x)$; sketch the asymptotes; plot the intercepts, maximum and minimum points, and inflection points. Make the curve pass through these points, rising and falling according to the information given by $f'(x)$, and according to the information given by $f''(x)$. If additional accuracy is desired near any point, you may compute the value of $f'(x)$ there, since they should be close enough; notice that the tangent indicates the direction in which $f(x)$ goes.

2.10 Slant asymptotes

Definition 2.6. A slant asymptote is a line $y = mx + b$ of a function $f(x)$ if and only if

$$\lim_{x \to \infty} (f(x) - (mx + b)) = 0.$$

This definition says that if the two curves — $f(x)$ and y — get closer and closer as $x \to \infty$, then y is a slant asymptote of $f(x)$. One way to compute where two curves get closer and closer is to find out whether the distance from the points in one curve to the other — that's what the definition does, in fact.

Perhaps the challenge is when to recognize that we have a slant asymptote. For example, we'll have slant asymptotes when the degree of the numerator of a rational function is one more than the degree of the denominator.

Exercise 2.37. Show that

$$f(x) = \frac{x^3}{x^2 + 1}$$

has a slant asymptote.

Solution. Perform long division to find that

$$f(x) = \frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}.$$

Notice that

$$-\frac{x}{x^2+1} \to 0 \text{ as } x \to \infty.$$

So $f(x) - x \to 0$ as $x \to \infty$, and $y = x$ is a slant asymptote of $f(x)$.

Exercise 2.38. Draw $f(x) = x^{2/3}(6-x)^{1/3}$.

Solution. We'll follow exactly what we defined above. We need the critical numbers, so we will find out where $f'(x)$ fails to exist, and where $f'(x) = 0$. Let $u = x^{2/3}, v = (6-x)^{1/3}$. So

$$\begin{aligned} f'(x) &= u'v + uv' \\ &= (2/3)x^{-1/3}(6-x)^{1/3} + x^{2/3}(-1/3)(6-x)^{-2/3} \\ &= \frac{4-x}{\sqrt[3]{x(6-x)^2}}. \end{aligned}$$

So we now know that the we have critical at $x = 0, x = 4$, and $x = 6$. Our intervals are

Now we know where it goes up and down, and so we have the local minima and the local maxima. We have a local minimum at $x = 0$ because it goes down in the negative side of x and goes up after touching $x = 0$. We have a local maximum at $x = 4$ because note that as it goes over $x = 4$, it changes from going up to going down.

Now, what does $f(x)$ look like? So far, I can say that it comes down from the left side and it will have a cusp at 0 because the f is continuous at $x = 0$, but not differentiable there — we have a vertical tangent there —, so that's a fair guess. From that point, it goes up and to

$$f(4) = (2 \cdot 2)^{2/3} 2^{1/3} = 2^{5/3} \approx 3.1748,$$

and it is differentiable at that $f(4)$, so I can say that it has a "smooth" high point at $y \approx 3.1748$, and then it comes down forever with a nondifferentiable point at $x = 6$ which gives us a vertical tangent there as well. Let's see if we can get more information.

Can $f''(x)$ gives us any more clues? Yes, $f''(x)$ can tell us about the concavities of $f(x)$. Computing $f''(x)$, we get

$$f''(x) = \frac{-8}{x^{4/3}(6-x)^{5/3}}.$$

Now we check where $f''(x)$ is positive and where it is negative. It is never zero, but it may be undefined at $x = 0$ and $x = 6$. Building a table, we see the following behavior.

So it's positive when $x > 6$, and negative to the left side. So f is concave down when $x < 0$, concave down when $0 < x < 6$ and concave up when $x > 6$. Now we're able to draw a accurate enough graph.

Exercise 2.39. Suppose you're given a formula for a function f. How can you tell where f is increasing or decreasing? How do you determine where the graph of f is concave upward or concave downward? How do you locate the inflection points?

Solution. First, look at f'. Break the real line in as many intervals as necessary given by the zeroes of f' and where f' is not defined — the critical numbers of f. If f' has n critical numbers, you get $n+1$ intervals. Pick a number in each of these intervals and notice where f' is positive and negative. If positive, then f is increasing, if negative, then f is decreasing.

Second, look at f''. Find the critical numbers of f'. Break the real line into as many intervals as required by the number of critical numbers of f'. Notice where it is positive, and where it is negative. Where it is positive, f is concave upward; otherwise, downward.

Third, using the second part, notice where it changes sign; we have an inflection point only at the points where it changes sign.

2.11 Optimizing things

If you have a function that describes something, then you can look for the absolute maximum and mininum and you'll have a clue of where in the input of the function you have a maximization or a minimization of the output. Finding minima and maxima values is actually not too hard, but finding the functions is sometimes tricky, as we should expect, because to describe phenomena in the universe requires considerable skills.

Exercise 2.40. Find two integers whose difference is 100 and whose product is a minimum.

Solution. Let x, y be integers with $x < y$. Then $y - x = 100$. We want to minimize $f(x) = xy$. We know that $y = 100 + x$. So $f(x) = x(100 + x)$ and $f'(x) = 2x + 100$. Now $f'(x)$ is zero when $x = -50$, and there $y = 150$.

So $150 - 50 = 100$ and the product is $150 \cdot 50 = 7500$. We're done. We could object to this solution saying that we're not sure whether $x = -50$ is a minimum, but possibly a local extremum. That's futile, though, because $f(x)$ is a parabola concave upward, so I know that where $f'(x) = 0$, we do have a local mininum.

Exercise 2.41. Find the point on the line $y = 4x + 7$ that is closest to the origin.

Solution. Here we're required to develop a distance function, and for the plane of Descartes, we have the famous

$$d = \sqrt{(a-0)^2 + (b-0)^2}.$$

We want the distance from a point on y curve to the origin. To find the distance, we need the point (a, b) which is common for both functions; that is

$$d = \sqrt{x^2 + (4x+7)^2}.$$

Now, differentiating d is harder than differentiating $d^2 = x^2 + (4x+7)^2$, so we will work with d^2 because they both share the same minimum — but notice that this is not true in general; for example, $f(x) = x^2 - 1$ and $f(x)^2 = (x^2-1)^2$ don't share minima. We have $(d^2)' = 2x + 8(4x+7)$ which has a zero when $x = -28/17$, so the point is $(-28/17, 7/17)$.

Exercise 2.42. Find the points on the hyperbola $y^2 - x^2 = 4$ that are closest to the point $(2, 0)$.

Solution. I'd like to see the hyperbola first. So I'm going to graph it. This hyperbola is symmetric about the x-axis and about the y-axis because $y^2 - x^2 = 4 = (-y)^2 - (-x)^2$. Let's consider just the $y > 0$ of the plane of Descartes.

We know that the curve touches $y = 2$ because if $x = 0$, then $y = \sqrt{4}$. As $x \to \infty$, $y \to \infty$, so there's no horizontal asymptote — nor any vertical one. There may be slant asymptotes.

We compute

$$y' = \frac{x}{\sqrt{4+x^2}},$$

which is zero when $x = 0$, so $x = 0$ is the only critical number. We compute

$$y'' = \frac{\sqrt{4+x^2} - x^2(4+x^2)^{-1/2}}{4+x^2},$$

which is never negative. From y' and y'', we conclude the following table.

So, there's a local minimum at 0, and no inflection points. What about a slant asymptote? If there's one, and it's $g(x)$, then it must be true that

$$\lim_{x\to\infty} \sqrt{4+x^2} - g(x) = 0.$$

Suspecting that $g(x) = x$ will do, we let $q(x) = \sqrt{4+x^2} + x$. Then

$$\begin{aligned}\sqrt{4+x^2} - x &= (\sqrt{4+x^2} - x) \cdot \frac{q(x)}{q(x)} \\ &= \frac{4 + x^2 - x^2}{\sqrt{4+x^2} + x} \\ &= \frac{4}{\sqrt{4+x^2} + x} \\ &= 0,\end{aligned}$$

as $x \to \infty$ because $(4+x^2) \to \infty$, and $\sqrt{4+x^2}$ as well, and certainly does x. So $g(x) = x$ is a slant asymptote. Now we have a pretty good idea of what it looks like. Let's begin then. The distance from $(2, 0)$ to the hyperbola is given by

$$d = \sqrt{(x-2)^2 + y^2}.$$

To work with only one variable, we must notice the points we are interested in considering are simply the points of the hyperbola, so we may replace y^2 in d by $4 + x^2$. So

$$d = \sqrt{(x-2)^2 + 4 + x^2}.$$

But it's easier to work with

$$d^2 = (x-2)^2 + 4 + x^2,$$

and we can since they share the same minimum. We now minimize d^2 by computing $(d^2)' = 4x - 4$, which is zero when $x = 1$. So the closest points to $(2, 0)$ are $(1, \sqrt{5})$ and $(1, -\sqrt{5})$.

Exercise 2.43. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.

Solution. The formula of a circle is $x^2 + y^2 = r^2$. We can position this circle at the origin of the plane of Descartes. Then we can work with $y = \sqrt{r^2 - x^2}$ which is half of the circle. Notice that the height of the rectangle will touch the circle — it's inscribed, and it only makes sense this way, since we want to maximize the area. So, if a rectangle has area $A(x, y) = xy$, then we will be working with

$$A(x) = 2x\sqrt{r^2 - x^2}.$$

Now we have a function of one variable which we wish to maximize. To maximize it, we take its derivative which is

$$A'(x) = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}}.$$

This function is zero when $x = \frac{\sqrt{2}r}{2}$. So the largest area is

$$A(x) = \sqrt{2}r\sqrt{r^2 - \frac{r^2}{2}}.$$

Exercise 2.44. Find the area of the largest isosceles triangle that can be inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution. Without computing too much, let's develop the algorithm required. The area of a triangle is

$$A(x, h) = \frac{xh}{2}.$$

In order to work with a single variable, we need h with respect to x. We know that one side of the triangle will have length $\sqrt{h^2 + x^2}$. We know that the height of the triangle will always touch the ellipse, so

$$h = y.$$

We can replace y by computing

$$\begin{aligned}
x^2 + \frac{a^2y^2}{b^2} &= a^2 \\
b^2x^2 + a^2y^2 &= b^2a^2 \\
a^2y^2 &= b^2a^2 - b^2x^2 \\
y^2 &= b^2 - \frac{b^2x^2}{a^2} \\
y &= \sqrt{b^2 - \frac{b^2x^2}{a^2}}.
\end{aligned}$$

Therefore,

$$A(x) = \frac{x\sqrt{b^2 - \frac{b^2x^2}{a^2}}}{2}.$$

Now maximize it to get all x such that $A'(x) = 0$. Hopefully, there's only one that makes sense. The answer will be with respect to the constants a and b. Plug the answer in $A(x)$ to get the largest area.

Hmm, nope. This solution is fallacious. The solution assumes that the triangle will only take the space possible in half of the ellipse. We may get a larger triangle by using the other half too; I don't know if that wouldn't happen, though. I don't know.

2.12 Applications to economics

Suppose that the cost of feeding x people in Brazil is described by

$$C(x) = 0.3x^3 - 2x^2 + 20.$$

This function has a local minimum at $x = 4 + 4/9$. We could, therefore, think that this is the exact point that we should produce — so that we can take advantage of low cost level, but as we shall see, there is another interesting point of this function which seems to be the an optimized way to produce with respect to cost per unit.

Let's consider the function

$$c(x) = C(x)/x,$$

which is called the average cost function. By dividing $C(x)$ by x, we are looking at the cost per unit. That is, if a supplier gives us a discount for buying a lot, we simply pretend that the price per unit is lower. Therefore, looking at $c(x)$, we would want to produce x units such that x is a local minimum of $c(x)$. Correct?

We see that the minimum of $c(x)$ is not the same minimum of $C(x)$. Why would it be? They are different functions. Also, let's take both minima and compute how much it would cost to produce x units in both functions.

The average cost function is

$$c(x) = C(x)/x = 0.3x^2 - 2x + 20/x$$

and it intersects the marginal cost function at $x \approx 4.78$. Now, let's compare. We have $C(4.78) \approx 7.06$, while $C(4+4/9) \approx 6.83$. But we have $c(4.78) \approx 1.47$, while $c(4+4/9) \approx 1.53$.

So, it is worth producing a little more in order to get a lower unit cost — even though the cost of production is increasing a bit. Perhaps because this function is a little unusual, it shows this behavior, or perhaps it doesn't make much sense otherwise.

Notice that picking a local minimum at the cost function doesn't make too much sense. Suppose, for instance, that the minimum shows up close to 0. Will us produce almost nothing just to get the best price? Probably not. Also, notice that $C(x)$ is a very unusual cost function; most cost functions don't have local minima, I'd say — it's very unusual to buy, say, 10 and pay 10, and then getting a cost of 5 for buying 20 from the same supplier.

2.13 Sigma notation

We're about to study about integrals. It will be useful, therefore, to look into sigma notation used in Riemann sums. If we want to write $2^3 + 3^3 + ... + n^3$, we pick an index i and we replace the sum by

$$\sum_{i=2}^{n} i^3.$$

The letter n used in the sum must not be confused with the letter n used in limit expressions which is seen in Riemann sums. Suppose that we're computing an approximation of the area under some curve f. We write

$$A \approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x.$$

Using n rectangles, we get

$$A \approx f(x_1)\Delta x + ... + f(x_n)\Delta x.$$

To write this as a sum, we let i be an iterator, and so i goes from 1 to n. That is

$$A \approx \sum_{i=1}^{n} f(x_i)\Delta x.$$

Now, if we want to increase n forever, we say so by writing

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

Notice now that we have an exact area. These sums are called Riemann sums because Bernhard Riemann was calculating "areas" this way, I think. It is indeed a very remarkable technique. What we call a Riemann sum, though, is always an approximation to an area; that's what we call it today. Perhaps Riemann was already taking limits to find exact areas.

By the way, the word "exact" used in conjunction with area is a mathematical exactness. We defined the area under a curve to be the integral of the curve over some interval — so, the function must be continuous on that interval. So it is exact because it is by definition anyway. The definition of an integral, as we shall see, is what we see above: the limit, as $n \to \infty$, of a Riemann sum with n rectangles.

3 Integrals

We start with the definition of a definite integral. Let's first observe what the definition is; let's look at its words and symbols, and after that, let's talk about ways of thinking of it.

Definition 3.1. Let f be a continuous function defined on $[a, b]$. Break this interval into n equal pieces so that each piece will have a length of $\Delta x = (b - a)/n$. Let $x_0, ..., x_n$ be the endpoints of these subintervals and we choose arbitrary points $x_1^*, ..., x_n^*$ in these subintervals an define the integral of f from a to b as

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_1^*)\Delta x.$$

When $f(x) \geq 0$ for all x in $[a, b]$, then a definite integral is an area; that is, a definite integral is a number; it signifies a quantity; an amount of squares of some unit such as meters. If that's not the case, then we may get a difference of areas, or perhaps a completely negative area. The latter case if when $f(x) \leq 0$ for all x in $[a, b]$. None of these cases is any problem; it's a matter of interpretation. Mathematics is not really concerned with these interpretation; mathematics is concerned with definitions and theorems, but I don't mean mathematicians. I believe that mathematicians are very much concerned with these interpretations, but "negative area" is not problem for a mathematician. If a negative balance in your account if money that you owe, then a negative area is an area that you don't have.

We talk about these arbitrary points in the definition because they are really arbitrary; that is, we can pick any points in these subintervals whatsoever. In cases when our function is a table of values, then we will of course pick the points in which we have values; a table of values is a very discontinuous function. What I mean is that we have a finite number of points in which the function is defined, so the function is not continuous anywhere. So we can't integrate it; but we can get an approximation, anyway; as Riemann used to do. In situations such as this one described, we are indeed picking arbitrary points. In an abstract case; in cases where we're dealing with mathematical functions, we will indeed choose equally spaced points if we're going to integrate directly through Riemann's technique. The truth is that for mathematical functions, we integrate using our integration methods, but numerical integration is always useful for when we don't have a continuous function to work with.

Some discontinuities can be resolved. For example, suppose that a function is continuous on an interval $[a, c)$ and continuous on an interval $(c, b]$. That is, the function is continuous on $[a, b]$ except at c. Suppose now that the limit of the function as we approach c exists and has value L. Then we may, simply... hah, I don't know if we can do what I'm thinking. For a non-abstract case, indeed; but for an abstract function, I don't know. What I had in mind was to patch the function, but if we patch it, we have now a continuous function and then of course we can integrate.

When we use Riemann's technique to integrate, we often pick the midpoints of the n subintervals we choose. That's because for some common functions, we actually get a pretty good approximation of the area by picking the midpoints.

3.1 Properties of integrals

In our definition of the definite integral, we assumed that the lower limit is a point on the x-axis which is more to the left of the upper limit, but a Riemann sum makes sense even if the lower limit is greater than the upper limit. Suppose we want to integrate over an interval $[a, b]$, but we want to start from the right and add to the left; then Δx changes from $(b-a)/n$ to $(a-b)/n$. So

$$\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx.$$

Also, if a $= b$, then $\Delta x = 0$ so

$$\int_a^a f(x)\ dx = 0.$$

Also, integrating a constant, is the same as computing the area of a rectangle because a constant function is a straight line. So, if the interval is $[a, b]$, then the integral will be equal to $c(b-a)$, where c is any constant. Also, the sum of two integrals is equal to the integral of the sum of two functions over the same interval.

You can always concatenate adjacent intervals as says the property

$$\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx.$$

If $f(x) \geq 0$ for $[a, b]$, then

$$\int_a^b f(x)\ dx \geq 0.$$

This makes total sense, no? Similarly, if $f(x) \geq g(x)$ for $[a, b]$, then

$$\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx.$$

Suppose now that $m \leq f(x) \leq M$ for $[a, b]$, where m and M are constants. Then, by using the property defined above, we can write

$$\int_a^b m\ dx \leq \int_a^b f(x)\ dx \leq \int_a^b M\ dx,$$

which in turn implies that

$$m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a).$$

which has a meaningful geometric interpretation. This property is very useful to estimate integrals. It also allows us to start a chain of reasoning if we're dealing with areas or integrals, since all integrals have this property.

Suppose for instance, that we want to estimate the area

$$\int_1^4 \sqrt{x}\ dx.$$

Since $\sqrt{x}$ is defined on $[1, 4]$, then we can use the "closed interval theorem" to get the absolute minimum and absolute maximum of the function in this interval. So it's true that

$$1(4-1) \leq \int_1^4 \sqrt{x}\ dx \leq 2(4-1)$$

$$3 \leq \int_1^4 \sqrt{x}\ dx \leq 6.$$

So the area we're looking for indeed lies between the 3 and 6 units square, or is 3 or 6 units square.

Let's estimate some integrals now, using Riemann sums. Suppose that $f(x) = 2 - x^2$ for $0 \leq x \leq 2$. We will take the right endpoints as heights for the rectangles. We have each $\Delta x = 2/4 = 1/2$. So our heights will be computed from $1/2, 2/2, 3/2, 4/2$. The ith height will be $i/2$ from $i = 1$ to $i = 4$. So

$$\sum_{i=1}^{4} \frac{1}{2}\left[2 - \left(\frac{i}{2}\right)^2\right] = \frac{1}{4}$$

Now, if we would use n rectangles, then we would get a $\Delta x = 2/n$. Each height will be computed from $2/n, 4/n, 6/n, ..., 2n/n$. So, we get a formula for a Riemann sum, which when taking the limit, we find

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[2 - \left(\frac{2i}{n}\right)^2\right] = \frac{4}{3} = 1 + \frac{3}{9}.$$

Suppose that $f(x) = 3x - 7$ for $0 \leq x \leq 3$. Let's compute the integral by using Riemann sums. We will break the interval down in n pieces, so each piece will be $3/n$. Using the right endpoints, we will compute the heights from $3/n, 6/n$, 9n, ..., 3n/n. In general, we will compute the heights from 3i/n from 1 to n. In symbols, we have

$$\lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \frac{9i}{n} = \frac{27}{2}$$

Suppose that $f(x) = x - 2\sin(2x)$ for $0 \leq x \leq 3$. In general, $\Delta x = 3/n$. So the heights will be computed from $3i/n$ from 1 to n. In symbols, we have the integral equal to

$$\lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \frac{3i}{n} - 2\sum_{i=1}^{n} \sin\left(\frac{6i}{n}\right)$$

I don't know how to compute the second sum. The sine function seems to complicate matters to me. I would, of course, start by looking for a pattern there, and rewrite the sine function as a new sum involving the pattern of values that I found by computing many sine function calls from the values that interests us; perhaps this way we would be able to get rid of the sine expression.

Let us now turn ourselves to the midpoint rule. Suppose that we want to compute the area under $2 - x^2$ for $0 \leq x \leq 2$ with 4 rectangles. Dividing a distance of 2 in 4 pieces will yield a $1/2$ distance for each piece. We are interested in the midpoints of each of these pieces, so we will start with the midpoint between 0 and $1/2$ which is $1/4$; the next will be $(1/2 + 1)/2 = 3/4$; the next will be $5/4$. In general, we have $(2i + 1)/4$ where i ranges from 0 to 3. Therefore, we have

$$\sum_{i=0}^{3} \frac{1}{2} \left[2 - \left(\frac{2i+1}{4} \right)^2 \right] = 1.375$$

which is a pretty good approximation, considering the amount of rectangles we're using.

Exercise 3.1. Estimate $\int_0^{10} \sin(\sqrt{x})\ dx$.

Solution. We will compute a Riemann sum with 5 rectangles by using midpoints to compute heights. So we write

$$\int_0^{10} \sin(\sqrt{x})\ dx \approx \sum_{i=0}^{4} 2(\sin(\sqrt{2i+1})) \approx 6.64.$$

Because the interval goes from 0 to 10, we chose to divide it in 5 pieces so that we can get simple looking expressions; we have, for example, $\Delta x = 2$, and the heights, computed from midpoints, are expressed 2i +1 where i goes from 0 to 4. The integral itself is approximately 6.28, so we have a pretty good approximation.

Exercise 3.2. Estimate $\int_1^2 \sqrt{1 + x^2}\ dx$.

Solution. We will compute a Riemann sum using 10 rectangles. We will have a $\Delta x = 1/10$. We will compute the heights by using the midpoints of each subinterval. The first left endpoint is 1, and the second is $1 + 1/10 = 11/10$. The second left endpoint is $11/10 + 1/10 = 12/10$, and it goes on like that. The first midpoint is $(1 + 11/10)/2 = 21/20$. The second midpoint is $(12/10 + 13/10)/2 = 25/20$.

It is not clear to me what the ith midpoint will be. We will then develop some algebra here to see if we can have a general form for midpoints, and then come back to solve this problem.

3.2 Interval patterns in Riemann sums

The length of each rectangle is usually computed by dividing the interval $[r, s]$ in n pieces. So $\Delta x = (s - r)/n$. Now, what is the first point of the interval?

It is certainly r. The next point will be $r + (s - r)/n$, and the next will be $r + 2(s - r)/n$, and the next will be $r + 3(s - r)/n$. Therefore, the ith will be $r + i(s - r)/n$. We are interested as well in the midpoints.

So we need to half of the sum of two adjacent points. That is,

$$\frac{\left[r + \frac{(i-1)(s-r)}{n}\right] + \left[r + \frac{i(s-r)}{n}\right]}{2}.$$

But notice that this is a formula for midpoints, not a pattern. We want a pattern as well, so that we can write a Riemann sum right away. In order to find a pattern, we will look at some values of i. When $i = 1$, we have

$$\frac{1}{2}\left(\left[r + 0 \cdot \frac{s-r}{n}\right] + \left[r + \frac{s-r}{n}\right]\right) = \frac{1}{2}\left(2r + \frac{s-r}{n}\right) = r + \frac{s-r}{2n}$$

and the sequence goes as

$$r + \frac{(s-r)}{2n}, r + \frac{3(s-r)}{2n}, r + \frac{5(s-r)}{2n}, ..., r + \frac{(2i+1)(s-r)}{2n},$$

where i starts at zero. Now, we must discover what is the last value of i. Would it be simply n? Or perhaps $n-1$? I say that we will have n midpoints; I also say that the first is obtainable by $i = 0$, so the last must be obtainable by $i = n-1$. No? I think so. We're now with enough tools to work on that previous problem.

Exercise 3.3. Estimate $\int_1^2 \sqrt{1+x^2}\ dx$.

Solution. We will compute a Riemann sum using 10 rectangles. We will have a $\Delta x = 1/10$. By the work we've done above, we will have the heights as

$$r + \frac{(2i+1)(s-r)}{2n}$$

where $r = 1, s = 2$, and $n = 10$. Then, we write the Riemann sum as

$$\frac{1}{10}\sum_{i=0}^{9}\sqrt{1 + \left(1 + \frac{2i+1}{20}\right)^2} \approx 1.81$$

Exercise 3.4. Evaluate $\int_{-1}^5 1 + 3x\ dx$.

Solution. We now want the integral really; but since we don't have techniques yet, we will compute it as Riemann used to do. By the work we developed above, and knowing that we're working with n subinterval of length $6/n$, our heights will be

$$r + \frac{(2i+1)(s-r)}{2n},$$

where $r = -1, s = 5$, so

$$-1 + \frac{3(2i+1)}{n},$$

and then we can compute

$$\lim_{n\to\infty}\frac{6}{n}\sum_{i=1}^{n}\left[1+3\left(-1+\frac{3(2i+1)}{n}\right)\right]=42,$$

Now, if we're suspicious of the formula above for the midpoints, let's pick the right endpoints which will be the following sequence $-1+6/n, -1+2\cdot 6/n, -1+3\cdot 6/n$, ..., $-1+6i/n$. So we write

$$\lim_{n\to\infty}\frac{6}{n}\sum_{i=1}^{n}\left[1+3\left(-1+\frac{6i}{n}\right)\right]=42,$$

and as a final check, let's carefully compute as

$$\begin{aligned}\int_{-1}^{5}1+3x\,dx &= \lim_{n\to\infty}\frac{6}{n}\sum_{i=1}^{n}\left[1+3\left(-1+\frac{6i}{n}\right)\right]\\ &= \lim_{n\to\infty}\frac{6}{n}\sum_{i=1}^{n}\left[1-3+\frac{18i}{n}\right]\\ &= \lim_{n\to\infty}\frac{6}{n}\sum_{i=1}^{n}-2+\frac{6}{n}\sum_{i=1}^{n}\frac{18i}{n}\\ &= \lim_{n\to\infty}\left(-12+\frac{60+48}{n^2}\sum_{i=1}^{n}i\right)\\ &= \lim_{n\to\infty}\left[-12+\frac{108}{n^2}\left(\frac{n(n+1)}{2}\right)\right]\\ &= \lim_{n\to\infty}\left[-12+108\left(\frac{1}{2}+\frac{1}{n}\right)\right]\\ &= -12+54=42.\end{aligned}$$

So perhaps we may now trust the pattern developed above for midpoints.

Exercise 3.5. Prove that

$$\int_0^{\pi/4}\sin^3 x\,dx \le \int_0^{\pi/4} sin^2 x\,dx.$$

Solution. We know that $\sin(x)\ge 0$ for all x in $[0,\pi/4]$. So $\sin^3(x)\ge 0$. Since $\sin(x)\le 1$ for all x real, then $\sin^3(x)\le 1$ for all x real. So $\sin^3(x)\le\sin^2(x)$ for all x in $[0,\pi/4]$. So

$$\int_0^{\pi/4}\sin^3 x\,dx \le \int_0^{\pi/4} sin^2 x\,dx.$$

I have to prove that $ab\le a$ if and only if $0\le b\le 1$. I don't know how to prove that right now.

Exercise 3.6. Prove that

$$\int_1^2 \sqrt{5-x}\ dx \geq \int_1^2 \sqrt{x+1}\ dx.$$

Solution. We must show that $f(x) = \sqrt{5-x} \geq g(x) = \sqrt{x+1}$ for all x in $[1, 2]$. We have $f(1) = 2 \geq g(1) = \sqrt{2}$. At $x = 2$, we have $f(2) = \sqrt{3} \geq \sqrt{3} = g(2)$. Now notice that

$$f'(x) = \frac{-1}{2\sqrt{5-x}} \text{ and } g(x) = \frac{1}{2\sqrt{x+1}}.$$

So $f(x)$ is an always decreasing function, while $g(x)$ is an always increasing function. This means that if they cross each other, they only do so once, and we will know that to the left of the crossing point, we have $f(x) \geq g(x)$, and to the right, we have $f(x) \leq g(x)$. This interesting point is $x = 3$. So, $f(x) \leq g(x)$ for all x in $[1, 2]$.

Exercise 3.7. Prove that

$$2 \leq \int_{-1}^1 \sqrt{1+x^2}\ dx \leq 2\sqrt{2}.$$

Solution. Let $f(x) = \sqrt{1+x^2}$. So $f(-1) = \sqrt{2}$. Notice that our domain is $-1 \leq x \leq 0$ union $0 \leq x \leq 1$. Start with $0 \leq x \leq 1$. Square it to get $0 \leq x^2 \leq 1$ which implies

$$1 \leq x^2 + 1 \leq 2.$$

Since every term of the inequality above is positive, we may take the square root of each of them to get

$$1 \leq \sqrt{x^2+1} \leq \sqrt{2}.$$

Now integrate that to get

$$\int_{-1}^2 1\ dx \leq \int_{-1}^1 \sqrt{x^2+1}\ dx \leq \int_{-1}^1 \sqrt{2}\ dx,$$

which implies

$$2 \leq \int_{-1}^1 \sqrt{x^2+1}\ dx \leq 2\sqrt{2}.$$

as desired.

Exercise 3.8. Prove that

$$\frac{\pi}{6} \leq \int_{\pi/6}^{\pi/2} \sin(x)\ dx \leq \frac{\pi}{3}.$$

Solution. We'll try constructing the result. The distance that we will integrate is $\pi/2 - \pi/6 = \pi/3$. We will work backwards to find which constants we should integrate. For instance, we need a constant m such that $m(\pi/2 - \pi/6) = \pi/6$;

so $m = 1/2$. Also, we need a constant M such that $M(\pi/2 - \pi/6) = \pi/3$; so $M = 1$. Now, consider the inequality

$$m = 1/2 \leq x \leq 1 = M.$$

We know that $\sin(\pi/6) = 1/2$ and that $\sin(\pi/2) = 1$. So it's true that $1/2 \leq \sin(x) \leq 1$ for all x in $[\pi/6, \pi/2]$. So we now integrate this inequality above to write

$$\int_{\pi/6}^{\pi/2} \frac{1}{2}\, dx \leq \int_{\pi/6}^{\pi/2} \sin(x)\, dx \leq \int_{\pi/6}^{\pi/2} 1\, dx$$
$$\frac{\pi}{6} \leq \int_{\pi/6}^{\pi/2} \sin(x)\, dx \leq \frac{\pi}{3}.$$

So we have a procedure here. We discovered the constants that would work by computing $m(u - l) = d$, where u is the upper limit and l is the lower limit, and d is the desired constant. We solve for m, and we're ready to set up the inequality of integrals, and integrate each term.

Exercise 3.9. Estimate

$$\int_1^2 \frac{1}{x}\, dx.$$

Solution. The method here is to look for the absolute minimum and maximum. We may use the "extreme value theorem" of continuous functions. Mininum and maximum, respectively, are, 1/2 and 1. So

$$\frac{1}{2}(2-1) \leq \int_1^2 \frac{1}{x}\, dx \leq 1(2-1).$$

Exercise 3.10. Estimate

$$\int_0^2 \sqrt{x^3+1}\, dx.$$

Solution. Minimum and maximum, respectively, are 1 and 3. So

$$2 \leq \int_0^2 \sqrt{x^3+1}\, dx \leq 6.$$

Exercise 3.11. Estimate

$$\int_{\pi/4}^{\pi/3} \cos(x)\, dx.$$

Solution. We have $\cos(\pi/4) = \sqrt{2}/2$, and $\cos(\pi/3) = 1/2$. So

$$\frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \leq \int_{\pi/4}^{\pi/3} \cos(x)\, dx \leq \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

which implies

$$\frac{\pi}{48} \leq \int_{\pi/4}^{\pi/3} \cos(x)\, dx \leq \frac{\pi}{24}.$$

Exercise 3.12. Estimate

$$\int_{\pi/4}^{3\pi/4} \sin^2(x)\ dx.$$

Solution. The minimum is $1/2$ at $\pi/4$ and $3\pi/4$, and the maximum is 1 at $\pi/2$. I'm afraid I need the property

$$\int_{\pi/4}^{3\pi/4} \sin^2 x\ dx = 2\int_{\pi/4}^{\pi/2} \sin^2 x\ dx.$$

I'm pretty confident that this is true, but I have no formal argument. I say that the derivative of $\sin^2(x)$ is $2\sin(x)\cos(x)$. Now I can prove, using the derivative, that from $\pi/4$ to $\pi/2$, the function is increasing, and decreasing from $\pi/2$ to $3\pi/4$. I can also prove that in these two intervals, the slope is only different by a sign. That is, in the same velocity that it goes up, it also goes down on the other interval. So, the area it covers is the same in both intervals. Therefore, the property above follows. Now, instead of estimating the left side of the equation, we estimate the right side. So we write

$$\frac{1}{4}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) \le \int_{\pi/4}^{\pi/2} \sin^2 x\ dx \le 1\left(\frac{\pi}{2}-\frac{\pi}{4}\right)$$

We multiply the inequality by 2 so that

$$\frac{\pi}{8} \le \int_{\pi/4}^{\pi/2} \sin^2 x\ dx + \int_{\pi/4}^{\pi/2} \sin^2 x\ dx \le \frac{\pi}{2}$$
$$\frac{\pi}{8} \le \int_{\pi/4}^{3\pi/4} \sin^2 x\ dx \le \frac{\pi}{2}.$$

The idea of looking for absolute minimum and absolute maximum helps quite a bit, but it will become troublesome as we work with not so well behaved functions; we may, nevertheless, break the badly behaved function into as many pieces as necessary and then look for minimum and maximum in these smaller closed intervals in which the function behaves somewhat well. So, as we can see, we cannot work with too complicated functions; only with somewhat bladly behaved ones.

Exercise 3.13. Prove that

$$\int_a^b x\ dx = \frac{b^2-a^2}{2}.$$

Solution. We start with the definition of the integral and *try* to find the result.

The definition says that

$$\begin{aligned}
\int_a^b x\ dx &= \lim_{n\to\infty} \sum_{i=1}^n \frac{b-a}{n}\left[a + \frac{i(b-a)}{n}\right] \\
&= \lim_{n\to\infty} \frac{b-a}{n}\left[\sum_{i=1}^n a + \sum_{i=1}^n \frac{i(b-a)}{n}\right] \\
&= \lim_{n\to\infty} \frac{b-a}{n}\left[an + \frac{b-a}{n}\frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty} a(b-a) + \lim_{n\to\infty}\left(\frac{b-a}{n}\right)^2 \frac{n(n+1)}{2} \\
&= a(b-a) + \lim_{n\to\infty} \frac{(b-a)^2}{2}\left(1+\frac{1}{n}\right) \\
&= a(b-a) + \frac{(b-a)^2}{2} \\
&= \frac{2a(b-a)+(b-a)^2}{2} \\
&= \frac{2ab-2a^2+b^2-2ab+a^2}{2} \\
&= \frac{-2a^2+b^2+a^2}{2} \\
&= \frac{b^2-a^2}{2}.
\end{aligned}$$

Exercise 3.14. Prove that

$$\int_1^3 \sqrt{x^4+1}\ dx \geq \frac{26}{3}.$$

Solution. I need the fact that

$$\int_a^b x^2\ dx = \frac{b^3-a^3}{3},$$

to prove the above. Assume so. Now notice that $f(x) = x^2$ has minimum and maximum, respectively, as 1 and 9 in $[1, 3]$. Also, we have $\sqrt{2}$ and $\sqrt{82}$ as a minimum and maximum, respectively, of $\sqrt{x^4+1}$. So, we know now that

$$\int_1^3 \sqrt{x^4+1}\ dx \geq \int_1^3 x^2\ dx = \frac{80}{3} \geq \frac{26}{3}.$$

Exercise 3.15. Prove that

$$\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx,$$

where c is a constant.

Solution. Straightforwadly, we write

$$\begin{aligned}\int_a^b cf(x)\ dx &= \lim_{n\to\infty}\sum_{i=1}^{n} cf(x_i)\Delta x\\ &= \lim_{n\to\infty} c\sum_{i=1}^{n} f(x_i)\Delta x\\ &= c\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x\\ &= c\int_a^b f(x)\ dx,\end{aligned}$$

where $\Delta x = (b-a)/n$.

Theorem 3.1. If $f(x)$ is continuous, then

$$\left|\int_a^b f(x)\ dx\right| \leq \int_a^b |f(x)|\ dx$$

Proof. The function $f(x)$ takes values on the interval $[a, b]$. At each of these values, the function gives us a real number. So we may use the property of real numbers to say that $-|f(x)| \leq f(x) \leq |f(x)|$, so we may write

$$\int_a^b -|f(x)|\ dx \leq \int_a^b f(x)\ dx \leq \int_a^b |f(x)|\ dx.$$

Notice now that $f(x)$ can yield negative and positive values, and $|f(x)|$ only positive values. So we may write

$$\left|\int_a^b f(x)\ dx\right| \leq \int_a^b |f(x)|\ dx.$$

In other words, if $f(x) \geq$ for all x in its domain, then we stricly have an equation; but if $f(x)$ takes a negative value anywhere in its domain, then we will stricly have an inequality. □

Exercise 3.16. Prove that

$$\left|\int_0^{2\pi} f(x)\sin(2x)\ dx\right| \leq \int_0^{2\pi} |f(x)|\ dx.$$

Solution. We now know that

$$\left|\int_0^{2\pi} f(x)\sin(2x)\ dx\right| \leq \int_0^{2\pi} |f(x)\sin(2x)|\ dx.$$

Since the integral is definite, it is a real number; so

$$\left|\int_0^{2\pi} f(x)\sin(2x)\ dx\right| \leq \int_0^{2\pi} |f(x)||\sin(2x)|\ dx.$$

Now notice that $|\sin(2x)| \leq 1$, so because of this, it follows that

$$\int_0^{2\pi} |f(x)||\sin(2x)| \; dx \leq \int_0^{2\pi} |f(x)| \; dx.$$

We actually haven't proved that $|a||b| \leq |a|$ when $|b| \leq 1$.

It is interesting to identify Riemann sums as integrals so that we can easily compute them with the fundamental theorem of calculus. The next exercise illustrates this.

Exercise 3.17. Compute

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5}.$$

Solution. In this case here, we may write

$$\lim_{n\to\infty} 1/n \sum_{i=1}^{n} \frac{i^4}{n^4},$$

and claim that the interval is from 0 to 1, and the integrand is x^4. In symbols,

$$\lim_{n\to\infty} 1/n \sum_{i=1}^{n} \frac{i^4}{n^4} = \int_0^1 x^4 \; dx.$$

As we have mentioned in previous pages, the sample points picked in a Riemann sum need not be right endpoints, left endpoints, or midpoints. We may pick whatever points in each subinterval of the Riemann sum to compute the height of the rectangle — as long as the $\Delta x \to 0$. The next exercise illustrates an unusual choice.

Exercise 3.18. Compute

$$\int_1^2 x^{-2} \; dx.$$

Hint: choose x_i^* to be the geometric mean of x_{i-1} and x_i. The geometric mean is the function $\sqrt{x_{i-1}x_i}$. Also, use the identity

$$\frac{1}{m(m+1)} = \frac{1}{m} - \frac{1}{m+1}.$$

Solution. Let $x_i = 1 + i/n$. This is our first endpoint. It is a right enpoint. By the hint, we will choose the geometric mean as the sample points. So let

$x_i^* = \sqrt{x_{i-1}x_i}$. So

$$\begin{aligned}
\int_1^2 x^{-2}\,dx &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\sqrt{\left[\left(1+\frac{i-1}{n}\right)\left(1+\frac{i}{n}\right)\right]^2}} \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\left(1+\frac{i-1}{n}\right)\left(1+\frac{i}{n}\right)} \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\left(\frac{n+i-1}{n}\right)\left(\frac{n+i}{n}\right)} \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{n^2}{(n+i-1)(n+i)} \\
&= \lim_{n\to\infty} n\sum_{i=1}^{n} \frac{1}{(n+i-1)(n+i)}.
\end{aligned}$$

Now let $m = n+i-1$ so that $m+1 = n+i$. By the hint, then, we have

$$\begin{aligned}
\lim_{n\to\infty} n\sum_{i=1}^{n}\left(\frac{1}{n+i-1} - \frac{1}{n+i}\right) &= \lim_{n\to\infty} n\sum_{i=1}^{n}\frac{1}{n+i-1} - n\sum_{i=1}^{n}\frac{1}{n+i} \\
&= \lim_{n\to\infty} n\sum_{i=0}^{n-1}\frac{1}{n+i} - n\sum_{i=1}^{n}\frac{1}{n+i} \\
&= \lim_{n\to\infty} n\left(\frac{1}{n} - \frac{1}{2n}\right) \\
&= \lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{1}{2} = \frac{1}{2}.
\end{aligned}$$

3.3 The fundamental theorem

The fundamental theorem of calculus is remarkable. The fundamental theorem makes two claims, and so we will break it down in two theorems.

Theorem 3.2. Let f be continuous on $[a, b]$. Define

$$g(x) = \int_a^x f(t)\,dt,$$

for $a \le x \le b$. Then $g(x)$ is continuous on $[a, b]$ and differentianble on (a, b) such that $g'(x) = f(x)$.

Proof. Let x be in (a, b). Let $x + h$ be in (a, b) with $h \neq 0$. Then

$$\begin{aligned} g(x+h) - g(x) &= \int_a^{x+h} f(t)\ dt - \int_a^x f(t)\ dt \\ &= \left(\int_a^x f(t)\ dt + \int_x^{x+h} f(t)\ dt\right) - \int_a^x f(t)\ dt \\ &= \int_a^x f(t)\ dt + - \int_a^x f(t)\ dt + \int_x^{x+h} f(t)\ dt \\ &= \int_x^{x+h} f(t)\ dt. \end{aligned}$$

Since h $\neq 0$, we may multiply this equation by $1/h$. So

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\ dt.$$

We now consider two cases; one in which $h > 0$ and one in which $h < 0$. First, let $h > 0$. Consider now the interval $[x, x+h]$. Since f is continuous, then by the extreme value theorem, there is a number u such that $f(u) = m$ is the absolute minimum of f in $[x, x + h]$. Similarly, there is a number v such that $f(v) = M$ is the absolute maximum of f in $[x, x + h]$. Therefore,

$$\begin{aligned} \int_x^{x+h} m\ dt \leq \int_x^{x+h} f(t)\ dt &\leq \int_x^{x+h} M\ dt \\ mh \leq \int_x^{x+h} f(t)\ dt &\leq Mh \\ f(u)h \leq \int_x^{x+h} f(t)\ dt &\leq f(v)h. \end{aligned}$$

Since $h > 0$, we may write

$$f(u) \leq \frac{1}{h}\int_x^{x+h} f(t)\ dt \leq f(v),$$

which is the same as

$$f(u) \leq \frac{g(x+h) - g(x)}{h} \leq f(v).$$

Now, if we let $h \to 0^+$, then we will be squeezing the interval $[x, x+h]$, and so we would expect that eventually $f(u) = f(x)$, and similarly for $f(v)$. In symbols,

$$\lim_{h\to 0^+} f(u) = \lim_{u\to x^+} f(u) = f(x).$$

Similarly for v, we have

$$\lim_{h\to 0^+} f(v) = \lim_{v\to x^+} f(v) = f(x).$$

So we may write

$$\begin{aligned} \lim_{h\to 0^+} f(u) \leq \lim_{h\to 0^+} &\frac{g(x+h) - g(x)}{h} \leq \lim_{h\to 0^+} f(v) \\ f(x) \leq \lim_{h\to 0^+} &\frac{g(x+h) - g(x)}{h} \leq f(x). \end{aligned}$$

Therefore, by the squeeze theorem, we have

$$\lim_{h\to 0^+} \frac{g(x+h)-g(x)}{h} = f(x).$$

Now let $h < 0$, and consider the interval $[x+h, x]$. Since f is continuous, then by the extreme value theorem, there is a number u in $[x+h, x]$ such that $f(u) = m$ is the absolute minimum of f in $[x+h, x]$. Similarly, there is a number v in $[x+h, x]$ such that $f(v) = M$ is the the absolute maximum of f in $[x+h, x]$. So,

$$\begin{aligned} \int_{x+h}^{x} m\ dt \leq \int_{x+h}^{x} f(t)\ dt &\leq \int_{x+h}^{x} M\ dt \\ mh \leq \int_{x+h}^{x} f(t)\ dt &\leq Mh \\ f(u)h \leq \int_{x+h}^{x} f(t)\ dt &\leq f(v)h. \end{aligned}$$

Since $h < 0$, we may write

$$f(u) \geq \frac{1}{h}\int_{x+h}^{x} f(t)\ dt \geq f(v),$$

which is the same as

$$f(v) \leq \frac{g(x+h)-g(x)}{h} \leq f(u).$$

Now, if we let $h \to 0^-$, then we will be squeezing the interval $[x+h, x]$, and so we would expect that eventually $f(v) = f(x)$, and similarly for $f(u)$. In symbols,

$$\lim_{h\to 0^-} f(v) = \lim_{v\to x^-} f(v) = f(x).$$

Similarly for u, we have

$$\lim_{h\to 0^-} f(u) = \lim_{u\to x^-} f(u) = f(x).$$

So we may write

$$\begin{aligned} \lim_{h\to 0^-} f(v) \leq \lim_{h\to 0^-} \frac{g(x+h)-g(x)}{h} &\leq \lim_{h\to 0^-} f(u) \\ f(x) \leq \lim_{h\to 0^-} \frac{g(x+h)-g(x)}{h} &\leq f(x). \end{aligned}$$

Therefore, by the squeeze theorem, we have

$$\lim_{h\to 0^-} \frac{g(x+h)-g(x)}{h} = f(x).$$

Together with our previous result, we have

$$\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f(x),$$

which shows that $f(x)$ is the derivative of $g(x)$, and therefore $g(x)$ is differentiable on (a, b) which implies that $g(x)$ is continuous on $[a, b]$. □

Theorem 3.3. If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\ dx = F(b) - F(a),$$

where F is any antiderivative of f; that is, F is any function such that $F'(x) = f(x)$.

Proof. Let

$$g(x) = \int_a^x f(t)\ dt.$$

We know that $g'(x) = f(x)$. That is, $g(x)$ is an antiderivative of $f(x)$. Suppose that $F(x)$ is any antiderivative of $f(x)$. We have a theorem which says that any two antiderivatives of a function $f(x)$ must differ only by a constant. So it is true that

$$\begin{aligned} F(x) - g(x) &= c \\ F(x) &= g(x) + c, \end{aligned}$$

for $a \leq x \leq b$. Both $F(x)$ and $g(x)$ are continuous on $[a,b]$. Since both are continuous, we may write

$$\begin{aligned} \lim_{x\to a^+} F(x) &= \lim_{x\to a^+} \Big(g(x) + c \Big) \\ F(a) &= g(a) + c. \end{aligned}$$

Similarly, as $x \to b^-$, we have

$$\begin{aligned} \lim_{x\to b^-} F(x) &= \lim_{x\to b^-} \Big(g(x) + c \Big) \\ F(b) &= g(b) + c. \end{aligned}$$

But notice that

$$g(a) = \int_a^a f(t)\ dt = 0 \text{ and } g(b) = \int_a^b f(t)\ dt,$$

and $g(b)$ describes the area under f from a to b which is what we're interested in. We then write

$$\begin{aligned} F(b) - F(a) &= (g(b) + c) - (g(a) + c) \\ &= g(b) + c - g(a) - c \\ &= g(b) + c - c \\ &= g(b). \end{aligned}$$

Therefore,

$$F(b) - F(a) = g(b) = \int_a^b f(t)\ dt,$$

as desired. □

So if we know the antiderivative of a function f, then we can integrate it easily using the second theorem of the fundamental theorem of calculus.

Outside mathematics, this theorem makes a lot of sense. If $v(t)$ is the velocity of an object and $s(t)$ is its position at a time t, then $s'(t) = v(t)$, so $s(t)$ is an antiderivative of $v(t)$. If we integrate $v(t)$, then what we're doing is computing the area under $v(t)$ which is the distance traveled. If the object moved from a to b, then $s(b) - s(a)$ is the distance traveled; that is, the integral of $v(t)$ from a to b. In symbols,

$$\int_a^b v(t)\ dt = s(b) - s(a).$$

Exercise 3.19. Differentiate

$$h(x) = \int_0^{x^2} \sqrt{1+r^3}\ dr.$$

Solution. Let $u = x^2$. Since $\mathrm{du}/\mathrm{dx} = 2x$ and since

$$\frac{dh}{dx} = \frac{dh}{du}\frac{du}{dx},$$

we have

$$\begin{aligned} h'(x) &= \frac{d}{dx}\int_0^u \sqrt{1+r^3}\ dr \\ &= \sqrt{1+u^3}\frac{du}{dx} \\ &= 2x\sqrt{1+x^6}. \end{aligned}$$

Exercise 3.20. Differentiate

$$h(x) = \int_2^{1/x} \sin^4(t)\ dt.$$

Solution. Let $u = 1/x$. Then

$$\begin{aligned} h'(x) &= \frac{d}{dx}\int_2^u \sin^4(t)\ dt \\ &= \sin^4(u)\frac{du}{dx} \\ &= -\frac{\sin^4(1/x)}{x^2}. \end{aligned}$$

Exercise 3.21. Differentiate

$$y = \int_3^{\sqrt{x}} \frac{\cos t}{t}\ dt.$$

Solution. Let $u = \sqrt{x}$. Then $du/dx = 1/2\sqrt{x}$. So

$$\begin{aligned} y' &= \frac{d}{dx}\int_3^u \frac{\cos t}{t}\,dt \\ &= \frac{\cos u}{u}\frac{du}{dx} \\ &= \frac{\cos\sqrt{x}}{2\sqrt{x}\sqrt{x}} \\ &= \frac{\cos\sqrt{x}}{2x}. \end{aligned}$$

Exercise 3.22. Convince yourself that

$$\int_{2x}^{3x} f(u)\,du = \int_{2x}^{0} f(u)\,du + \int_{0}^{3x} f(u)\,du.$$

Solution. Suppose $x = 1$. Then we have

$$\begin{aligned} \int_2^3 f(u)\,du &= \int_2^0 f(u)\,du + \int_0^3 f(u)\,du \\ &= -\int_0^2 f(u)\,du + \int_0^3 f(u)\,du, \end{aligned}$$

which does seem to be true.

Exercise 3.23. Find a function f and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x},$$

for all $x > 0$.

Solution. The fundamental theorem gives us an easy differentiation of the integral; so, we differentiate both sides of the equation to get

$$\begin{aligned} \frac{f(x)}{x^2} &= \frac{1}{x^{1/2}} \\ f(x) &= x^2 \cdot x^{-1/2} \\ f(x) &= x^{3/2}. \end{aligned}$$

So this a function that seems to be the correct one. The problem here is that we also must find a because it is not true in general. How would we find a? We assume we have found it already. Call it a. So there is such number; it is a. Since it exists, we may compute

$$6 + \int_a^a \frac{f(t)}{t^2}\,dt = 2\sqrt{a},$$

and it's easy to do so because we know the integral is zero. So

$$\begin{aligned} 6 + 0 &= 2\sqrt{a} \\ 3 &= \sqrt{a} \\ 9 &= a. \end{aligned}$$

It turns out that 9 is a unique solution to this problem.

Exercise 3.24. Suppose h is a function such that $h(1) = -2, h'(1) = 2, h''(1) = 3, h(2) = 6, h'(2) = 5, h''(2) = 13$ and h'' is continuous everywhere. Evaluate

$$\int_1^2 h''(u)du.$$

Solution. We know that

$$\int_1^2 h''(u)du = h'(u)\Big|_1^2 = 5 - 3 = 2,$$

don't we? So, we're done. This problem presents some superfluous information; possibly to confuse us.

3.4 The substitution theorem

Theorem 3.4. If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

Proof. Consider a function $F(g(x))$, where $F' = f$. By the chain rule, we know that the derivative of $F(g(x))$ is $F'(g(x))g'(x) = f(g(x))g'(x)$, so if we let $u = g(x)$, then

$$\begin{aligned}\int F'(g(x))g'(x)\ dx &= F(g(x)) + c \\ &= F(u) + c \\ &= \int F'(u)\ du \\ &= \int f(u)\ du,\end{aligned}$$

as desired. □

3.5 Volumes

We define volume as the limit of the sums of the areas of slices of the solid where each slice has a width of Δx and $\Delta x \to 0$. That is,

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\ dx.$$

4 Studying functions

We're going to look into a series of topics now; we are mostly using Calculus to study functions, however. We need to study about inverse functions first; then we will look into exponential functions, and slowly and scatteredly we will be looking into some interesting functions.

4.1 On inverse functions

If $f(b) = a$, then $f^{-1}(a) = b$. What would $f'(b)$ be? We want to express $f'(b)$ with respect to $(f^{-1})'$. For that, we write a theorem.

Theorem 4.1. Let f be one-to-one and differentiable with $g = f^{-1}$ and $f'(g(a)) \neq 0$. Then g is differentiable at a and

$$g'(x) = \frac{1}{f'(g(x))},$$

where $g = f^{-1}$.

Proof. Let $y = g(x) = f^{-1}(x)$. Then $f(y) = x$. Now,

$$\begin{aligned} f'(y)y' &= 1 \\ y' &= \frac{1}{f'(y)} \\ g'(x) &= \frac{1}{f'(g(x))}, \end{aligned}$$

as desired. □

Now, let us define exponential functions. Exponential functions are a^x where a is some positive number, usually $a > 1$. If a is between 0 and 1, then the function exponentially decays. Exponential functions are defined for all reals, but then one wonders what 2^π means. It means, by definition,

$$a^x = \lim_{r \to x} a^r,$$

for some rational r. So to answer the question, 2^π is the limit of the sequence of numbers $2^3, 2^{3.1}, 2^{3.14}$, and so on. In fact, this explains why the base of an exponential function cannot be negative. If it were, then irrational numbers could not be powers of these bases. If they were, complex numbers would arise. For instance, consider $(-2)^\pi$; the limit doesn't converge because when r is 3.1, the answer is not real; when r is 3.14, the answer is real; when r is 3.141, the answer is not real. It depends on the evenness or oddness of the last digit of r. So it does not converge to any real value; so the value is undefined.

4.2 The number e

Here's one way to define e. Consider 2^x and 3^x. Take the derivative of them. We get, for $f(x) = 2^x$,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h} \\ &= \lim_{h \to 0} \frac{2^x 2^h - 2^x}{h} \\ &= \lim_{h \to 0} \frac{2^x(2^h - 1)}{h} \\ &= 2^x \lim_{h \to 0} \frac{2^h - 1}{h}. \end{aligned}$$

For $g(x) = 3^x$, we have

$$g'(x) = 3^x \lim_{h\to 0} \frac{3^h - 1}{h}.$$

So the difficulty here is to find the limit that we left above. But what's the derivative of 2^x and 3^x at $x = 0$? It's

$$\begin{aligned} f'(0) &= 2^0 \lim_{h\to 0} \frac{2^h - 1}{h} \\ &= \lim_{h\to 0} \frac{2^h - 1}{h}. \end{aligned}$$

And for 3^x? It's

$$\begin{aligned} g'(0) &= 3^0 \lim_{h\to 0} \frac{3^h - 1}{h} \\ &= \lim_{h\to 0} \frac{3^h - 1}{h}. \end{aligned}$$

That is, $f'(x) = f'(0)2^x$, and $g'(x) = g'(0)3^x$. Observe, then, that the derivative of these exponential functions are themselves times a constant. What would this constant be? By letting $h \to 0$, we see that $f'(0) \approx 0.693$ and $g'(0) \approx 1.09$. So we wonder if there is a number p between 2 and 3 such that $h(x) = p^x$ has $h'(0) = 1$. There is such number; we call it e. That is, e is the number such that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

Now, consider the exponential function $f(x) = a^x$ where $a = e$. It has the remarkable property that $f'(x) = f'(0)a^x. = 1 \cdot a^x = a^x$. And, because of this interesting derivative, it also follows that

$$\int e^x \, dx = e^x + c.$$

Exercise 4.1. Find a root of $f(x) = e^x + x = 0$.

Solution. We'll use Newton's method. We know that $f(0) = 1$, and $f(-1) = 1/e - 1 < 0$. So, there's a root between 0 and 1. We'll initiate Newton's iteraction with $a = 1/2$. We may write

$$\begin{aligned} f'(a) &= \frac{y - f(a)}{x - a} \\ (x - a)f'(a) &= y - f(a) \\ xf'(a) - af'(a) &= y - f(a) \\ xf'(a) &= y - f(a) + af'(a). \end{aligned}$$

But, we're looking for what x gives $y = 0$. So we set $y = 0$, and solve for x. So

$$x = \frac{af'(a) - f(a)}{f'(a)}$$

After we compute x for $a = 1/2$, we would feed this value back into this equation, and again and again; if the value seems to converge, we then jump to the conclusion that we're pretty close to the root. We find that the value seems to converge to -0.56714329, so we take that as close enough to the root.

4.3 Logarithmic functions

How is $y = \log_a x$ defined? What we want it to mean is y being the number such that $a^y = x$. So $y = \log_a x$ is defined as $y = \log_a x$ if and only if $a^y = x$.

What would its domain be? They would be positive real numbers, so zero is not there. But why not? Let's see: $a^y = 0$ is not satisfied by any base a. We could try $a = 0$ and $y = 0$, but 0^0 is actually undefined because there seems to be no grounds for the convergence of this expression. But why not negative numbers? See the section on exponential and their inverses; the exponential functions do not have negative range, and so the logarithmic ones don't have negative domain — after all, one is the inverse of the other.

What would its range be? We know that for $\log_a x = y$, we have a is positive and x is positive. So what values for y are possible? Consider x large. Then y must be large too. Probably, then, $y \to \infty$ as $x \to \infty$. Now consider x small; for example, $a^y = 0.1$. Then y will have to be negative so that, for $p = -y$,

$$\frac{1}{a^p} = 0.1 = \frac{1}{10},$$

so $a^p = 10$. Therefore, between 0 and 1, $\log_a x < 0$. So the range is $(-\infty, \infty)$. The domain of a^x is all real numbers, so naturally the range of $\log_a x$ is all real numbers.

If $a = e$, then $\log_e x$ is called the natural logarithm function, and we write $\ln x$. This function is so used in mathematics that some authors prefer to write $\log x$ to mean the natural logarithm. Others prefer to let $\log x$ mean $\log_{10} x$ or $\log_2 x$. I would prefer $\log x$ to mean the natural logarithm because to me it is the most interesting one, and I use it a lot more often then the others.

I will stick to $\ln x$ because many authors do so, and I actually type one letter less than I would with $\log x$, so I will let that be my choice. Now, notice that although we write $\ln x$ instead of $\ln(x)$, it still is a function, and its inverse is e^x; so $\ln e^x = x$.

We will later define these functions again, in some other way, which seems to be more interesting, and some facts about these functions will be clarified.

Theorem 4.2. For a $\neq 1$, a > 0, it is true that

$$\log_a x = \frac{\ln x}{\ln a}.$$

Proof. Writing $\log_a x$ means to say that there is a number y such that $a^y = x$. Now take natural logarithm of both sides of this equation to get

$$\begin{aligned} y \ln a &= \ln x \\ y &= \frac{\ln x}{\ln a}, \end{aligned}$$

as we wanted. □

This proof shows that we can rewrite any logarithmic function in terms of $\ln x$. But also, notice that we can rewrite any logarithmic function in terms of the logarithm of any valid base. A machine, therefore, needs only to implement an algorithm for one base, and rewrite all the others in terms of the chosen base.

A quick note is that we have not shown why it's true that $\ln a^y = y \ln a$. Let us do that now; let's prove the theorems of logarithms. We're going to need the derivative of $\ln x$, so let's show it first.

Theorem 4.3. The derivative of $\ln x$ is $1/x$.

Proof. Let $y = \ln x$. So

$$\begin{aligned} y &= \ln x \\ e^y &= x \\ e^y y' &= 1 \\ y' &= \frac{1}{e^y} \\ &= \frac{1}{e^{\ln x}} \\ &= \frac{1}{x}. \end{aligned}$$

□

4.3.1 The theorems of logarithms

There are three important theorems; the product theorem, the division theorem and the power theorem. The product theorem lays a foundation for the other two, so we prove it first. The proof of the first comes out by considering a function $f(x) = \ln ax$. We differentiate it and we obtain a function equal to the $g(x) = \ln x$. If two functions have the same derivative, they must differ by a constant.

This fact is an important theorem of calculus which is a corollary of Fermat's theorem. Fermat's theorem is proved using another important theorem: the mean value theorem. The mean value theorem can be generalized and it was; by Cauchy. Cauchy's generalization is used to prove another important theorem; l'Hospital's theorem.

Theorem 4.4. For $x > 0, y > 0$, it's true that $\ln xy = \ln x + \ln y$.

Proof. Let $f(x) = \ln yx$. Notice, however, that y is a real number and x is a variable. Now differentiate f to get

$$\begin{aligned} f'(x) &= \frac{y}{yx} \\ &= \frac{1}{x} \end{aligned}$$

So f and $\ln x$ have the same derivative; so $\ln x - \ln yx = c$. Now, if the theorem is true, then $c = -\ln y$. We cannot assume any value for y because y is an arbitrary constant, but x is variable and takes all values of the domain of $\ln x$. Since $x = 1$ is in the domain, we may consider the equation when $x = 1$. We get

$$0 - \ln y = c,$$

as expected. Therefore $\ln x - \ln y = \ln yx$. □

Now we use this result to prove the others.

Theorem 4.5. For $x > 0$, it's true that $\ln x^y = y \ln x$.

Proof. We have that $x^y = x_1 x_2 \, ... \, x_y$. Apply the product theorem as much as necessary to write $\ln x^y = \ln x_1 \ln x_2 ... \ln x_y = y \ln x$. This isn't a formal argument, however. Perhaps we can simply prove it by induction.

Consider $\ln x^n$, where n is a natural number. Show that it works for $n = 1$, as it clearly does. Assume it works for some natural $k : \ln x^k = k \ln x$. Then notice that

$$\begin{aligned} \ln x^{k+1} &= \ln x^k x \\ &= \ln x^k + \ln x \\ &= k \ln x + \ln x \\ &= (k+1) \ln x. \end{aligned}$$

Therefore, it's true for all naturals. I guess we can similarly show for all negative integers, and take the case where $n = 0$ for granted. What about the rest of the reals, though? □

By the way, do not mistake $\ln(x^y)$ for $(\ln x)^y$; only the former has the property above. When I write without parentesis, I mean the first.

Theorem 4.6. For $x > 0, y \neq 0$, it's true that $\ln x/y = \ln x - \ln y$.

Proof. We know that $\ln x/y = \ln xy^{-1}$. So

$$\begin{aligned} \ln xy^{-1} &= \ln x + \ln y^{-1} \\ &= \ln x - \ln y, \end{aligned}$$

as desired. □

4.3.2 The prime number theorem

Let $\pi(n)$ be the number of primes less or equal to n. So $\pi(15) = 6$ because $2, 3, 5, 7, 11, 13$ are the 6 primes before 15. Also, we find that $\pi(25) = 9$, and $\pi(100) = 25$. There is a theorem, called the Prime Number Theorem, which asserts that

$$\lim_{n \to \infty} \frac{\pi(n)}{n \ln n} = 1.$$

Exercise 4.2. Find the inverse of

$$y = \frac{e^x + 1}{-e^x + 1}.$$

Solution. Try to group the exponential terms together, and factor them.

$$\begin{aligned} y &= \frac{e^x + 1}{-e^x + 1} \\ y - ye^x &= e^x + 1 \\ -ye^x - e^x &= 1 - y \\ e^x(-y - 1) &= 1 - y \\ e^x &= \frac{1 - y}{-y - 1} \\ x &= \ln \frac{1 - y}{-y - 1} \\ x &= \ln -\frac{1 - y}{y + 1} \\ x &= \ln \frac{1 + y}{1 - y}. \end{aligned}$$

It works.

Now, let us look into another technique. Any function $f(x) = g(x)^{h(x)}$ where $g(x) > 0$ can be analyzed as a power of e because we can write $g(x)$ as $e^{\ln g(x)}$ so that $f(x) = h(x)e^{\ln g(x)}$. This allows us to compute, for instance,

$$\lim_{x \to \infty} x^{\ln x}.$$

To compute, we write

$$\begin{aligned} \lim_{x \to \infty} x^{\ln x} &= \lim_{x \to \infty} e^{\ln x^{\ln x}} \\ &= \lim_{x \to \infty} e^{\ln x \ln x} \\ &= \lim_{x \to \infty} e^{(\ln x)^2} \\ &= \lim_{x \to \infty} e^{2 \ln x} = \infty. \end{aligned}$$

We know that $\ln x \to \infty$ as $x \to \infty$ and $e^x \to \infty$ as $x \to \infty$. So the limit above goes to ∞ too. An interesting observation is that we did not fallaciously used the property of logarithms to write the last step. We use a property of exponents; notice that $(a^x)^y = a^{xy}$.

Exercise 4.3. Find $f^{(n)}$ if $f(x) = \ln(x - 1)$.

Solution. The first derivative is

$$f'(x) = \frac{1}{x - 1}.$$

The second is

$$f''(x) = -\frac{1}{(x - 1)^2}.$$

The third is

$$f^{(3)}(x) = \frac{2}{(x-1)^3}.$$

The forth is

$$f^{(4)}(x) = -\frac{6}{(x-1)^4}.$$

Let us let n stand for the power of the denominator because it matches it straightforwardly. For the sign of the expression, let $(-1)^{n+1}$ determine the sign, because it is negative when n is even, and positive otherwise. So far, then, we have

$$f^{(n)} = \frac{(-1)^{n+1}k}{(x-1)^n}.$$

We have to figure out the term that will replace k more precisely. For k, we have that it goes as $1, 1, 2, 6, 24, \ldots$ so we may guess that they can be expressed as $n!$ with $n = 0, 1, 2, \ldots$ So we need match this list with respect to the n as we have already established in the formula above. For the first derivative, $n = 1$, and $k = 0!$; when $n = 2, k = 1!$; when $n = 3, k = 2!$. So we jump to the conclusion that

$$f^{(n)} = \frac{(-1)^{n+1}(n-1)!}{(x-1)^n}.$$

And we're done.

4.4 The general antiderivative of $1/x$

We know that $\ln x + c$ has $1/x$ as its derivative, but when we integrate $1/x$, we may have negative numbers as limits of integration and $\ln x + c$ is only defined for positive numbers. But it's valid to ask about the area "under $1/x$" on the negative side since $1/x$ is defined everywhere but at zero. So for limits on the negative side, the antiderivative $\ln x + c$ will not be helpful.

Consider $\ln|x| + c$. To differentiate this function, we consider $x > 0$, and $x < 0$. When $x > 0$, $\ln|x| + c = \ln x + c$, so the derivative is $1/x$. When $x < 0$, then $\ln|x| + c = \ln -x + c$, so the derivative is $(-1)/(-x) = 1/x$. Therefore,

$$\frac{d}{dx}\ln|x| = \begin{cases} 1/x & \text{if } x > 0, \\ 1/x & \text{if } x < 0. \end{cases}$$

In other words, the general antiderivative of $1/x$ is $\ln|x| + c$.

4.5 A nice definition of $\ln x$

Some people prefer to define $\ln x$ not as the inverse of exponential function e^x, but rather as the area under $1/x$. This definition is really nice because we can nicely show that it is the inverse of the exponential function, and it becomes quite apparent how to compute $\ln x$ for any $x > 0$. Let us define it then.

Definition 4.1. Define

$$\ln x = \int_1^x \frac{1}{t}\ dt,$$

for all $x > 0$.

The derivative of $\ln x$ is trivially obtained by the fundamental theorem. Let us now consider the inverse function of $\ln x$. We will call it $\exp x$, but it is not to be assumed yet that $\exp x = e^x$. We will show so.

Definition 4.2. Define

$$\exp x = y \iff \ln y = x.$$

Now, because they are inverse functions, we can write $\exp \ln x = x$ and $\ln \exp x = x$. Also, we have identities such as $\exp 0 = 1$ because $\ln 1 = 0$. We have $\exp 1 = e$ because $\ln e = 1$. Now if r is any rational number, then consider

$$\ln e^r = r \ln e = r.$$

Therefore, $\exp r = e^r$ because $\ln e^r = r$. This is how the exponential function comes to life as the inverse function of $\ln x$. We have not considered irrational numbers, however.

Let r be irrational. Consider a^r, where $a > 0$. We define

$$a^r = \lim_{x \to r} a^x.$$

Now consider $\ln e^r$. Since r is irrational, we have

$$\begin{aligned} \ln e^r &= \ln \lim_{x \to r} e^x \\ &= \lim_{x \to r} \ln e^x \\ &= \lim_{x \to r} x \ln e \\ &= \lim_{x \to r} x \\ &= r. \end{aligned}$$

But notice that we had trouble proving that the power theorem of logarithms works for all real numbers. Here we're assuming so. We assumed that $\ln x$ is continuous when we pulled the limit out from the $\ln x$ function.

4.6 Other exponential functions

Let us now define all the other exponential functions; we will define them in terms of the exponential function e^x.

Definition 4.3. Let $m > 0$. Define

$$m^x = e^{\ln a^x} = e^{x \ln a},$$

for all x.

This definition makes some proofs easy to be written. The trouble here is to trust that this definition makes sense. For instance, how do we know that $2^2 = e^{2\ln 2}$?

Theorem 4.7. Let m be a real number. Then $m^{x+y} = m^x m^y$.

Proof. Apply the definition above in

$$\begin{aligned} m^{x+y} &= e^{\ln m^{x+y}} \\ &= e^{(x+y)\ln m} \\ &= e^{x\ln m + y\ln m} \\ &= e^{x\ln m} e^{y\ln m} \\ &= m^x m^y, \end{aligned}$$

as desired. □

4.7 The inverse trigonometric functions

Now we will use the knowledge acquired so far to look at new functions, their properties and possibly some applications of them. For instance, we will begin to finding the derivative of the inverse trigonometric functions.

Let $y = \arcsin x$, so $\sin y = x$. The domain of $\arcsin x$ is $[-1, 1]$, and the range of $\arcsin x$ is $-\pi/2 \le y \le \pi/2$. Implicitly, we differentiate it to get

$$\begin{aligned} \cos y y' &= 1 \\ y' &= \frac{1}{\cos y}. \end{aligned}$$

With the range above, we have that $\cos y \ge 0$. We know that $\cos^2 y + \sin^2 y = 1$, so $\cos^2 y = 1 - \sin^2 y$, so $\cos y = \sqrt{1 - \sin^2 y}$. But notice that $\sin y = x$, so $\sin^2 y = x^2$. Therefore, $\cos y = \sqrt{1 - x^2}$. So

$$y' = \frac{1}{\sqrt{1 - x^2}}.$$

We perform the same strategy for $\arccos x$. The function $\cos x$ is one-to-one from 0 to π. So the domain of $\arccos x$ is $[-1, 1]$; the range is $[0, \pi]$. Let y = $\arccos x$, so $\cos y = x$. Then

$$\begin{aligned} -\sin y y' &= 1 \\ y' &= -\frac{1}{\sin y}. \end{aligned}$$

Since $\sin y = \sqrt{1 - \cos^2 y}$ and $\cos^2 y = x^2$, we have

$$y' = -\frac{1}{\sqrt{1 - x^2}}.$$

Let us now look into the inverse tangent function. We have to restrict $\arctan x$ on $(-\pi/2, \pi/2)$ because we need an one-to-one function. The domain for $\arctan x$ is $(-\infty, \infty)$ and range is $(-\pi/2, \pi/2)$. To find its derivative, we write

$$\begin{aligned} \arctan x &= y \\ x &= \tan y. \end{aligned}$$

Now differentiate with respect to x. We get

$$\begin{aligned} 1 &= \sec^2 y y' \\ y' &= \frac{1}{\sec^2 y}. \end{aligned}$$

We wish now to express $\sec^2 y$ in terms of x. Notice that $\sec^2 y = \tan^2 y + 1$. Since $x = \tan y$, then $\sec^2 y = x^2 + 1$. Therefore

$$y' = \frac{1}{x^2 + 1},$$

and we're done.

Now we know a few antiderivatives. For instance,

$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x + c,$$

and

$$\int -\frac{1}{\sqrt{1-x^2}}\, dx = \arccos x + c,$$

and

$$\int \frac{1}{x^2+1}\, dx = \arctan x + c.$$

Notice that sometimes we have integrands similar, though not exactly one of the above; but it's a matter of algebraically working it out for that to be so. For example,

$$\begin{aligned} \int \frac{1}{x^2 + a^2}\, dx &= \int \frac{dx}{a^2\left(\dfrac{x^2}{a^2} + 1\right)} \\ &= \frac{1}{a^2} \int \frac{dx}{\left(\dfrac{x}{a}\right)^2 + 1}. \end{aligned}$$

Now let $u = x/a$, so $dx = a\ du$. Then

$$\begin{aligned} \frac{1}{a} \int \frac{du}{u^2+1} &= \frac{1}{a} \arctan u + c \\ &= \frac{1}{a} \arctan \frac{x}{a} + c. \end{aligned}$$

A similar work can be done for all other inverse trigonometric functions.

Exercise 4.4. Evaluate
$$\int \frac{x\ dx}{x^4+9}.$$

Solution. First, factor the number 9 from the denominator. We get
$$\frac{1}{9}\int \frac{x\ dx}{\left(\frac{x^4}{9}+1\right)}.$$

Now let $u=(x^2)/3$, so that $dx=(3/2x)du$. Then we write
$$\begin{aligned}\frac{1}{9}\frac{3}{2}\int \frac{du}{u^2+1} &= \frac{1}{6}\arctan u + c\\ &= \frac{1}{6}\arctan\frac{x^2}{3}+c,\end{aligned}$$

as desired.

Exercise 4.5. Simplify $\cos(\arctan x)$.

Solution. Let $y=\arctan x$ so that $\tan y = x$. If $\tan y = x$, then thinking of a right triangle, we have an angle y and y's opposite side has length x, while y's adjacent side has length 1. Therefore, the hypothenuse has length $\sqrt{1+x^2}$, and $\cos y = 1/\sqrt{1+x^2}$. So
$$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$$

is simplified as desired.

Exercise 4.6. Prove that $\arctan x + \operatorname{arccot} x = \pi/2$.

Solution. Let $f(x) = \arctan x + \operatorname{arccot} x$. Notice that
$$f'(x) = \frac{1}{x^2+1} - \frac{1}{x^2+1} = 0.$$

This means that $\arctan x$ has the same derivative as $\operatorname{arccot} x$. Therefore, they must differ by a constant. That is, $f(x)=c$, for all x. Notice that
$$\arctan 1 + \operatorname{arccot} 1 = \frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}.$$

So $f(x) = \arctan x + \operatorname{arccot} x = \pi/2$.

Exercise 4.7. Find $\arcsin \sqrt{3}/2$.

Solution. We know that $\sin \pi/3 = \sqrt{3}/2$, so $\arcsin\sqrt{3}/2 = \pi/3$.

Exercise 4.8. Draw the following diagram: mark a point A and draw a straight line to the right up to a point P; continue the straight line up to a point B. From B draw a perpendicular line going up to a point Q. From Q draw a line that meets the point P. This procedure creates a right triangle where the angle PBQ

is a right angle. Now, from A, draw a perpendicular line going up to a point R. From R draw a line that meets the point P; so now that we have two right triangles where P is a point that they have in common. Now, let $AB = 3$ units. Let $RA = 5$ units, and let $QB = 2$ units. Let the angle RPA be called γ and let the angle QPB be called α. There's an angle β such that $\beta = \pi/2 - (\gamma + \alpha)$. Assuming we can move the point P between A and B adjusting the size of the hypothenuses of the triangles, how far from A should we position P as to maximize the angle β?

Solution. Let $PB = x$ so that $AP = 3 - x$. So $\mathrm{RP} = \sqrt{25 + (3-x)^2}$, and $\mathrm{PQ} = \sqrt{4 + x^2}$. Now, we need to express β with respect to x. So, naturally we will use trigonometric functions; but the ones that express angles are the inverse trigonometric ones, so we must find out which ones will give us a simple enough expression. Also, notice that β is maximized, when the sum $\gamma + \alpha$ is minimized, so we may attack the problem either way; I choose the latter; I'm going to minimize the sum $\gamma + \alpha$.

Notice that

$$\tan\gamma = \frac{5}{3-x}$$
$$\tan\alpha = \frac{2}{x}.$$

So, it's certainly true that

$$\gamma = \arctan\frac{5}{3-x}$$
$$\alpha = \arctan\frac{2}{x}.$$

Therefore

$$\gamma + \alpha = \arctan\frac{5}{3-x} + \arctan\frac{2}{x}.$$

Now the work is to differentiate this function, and find its local minimums, and identify which one is the one we are looking for. Notice that because $AB = 3$ units, P can only range from A to B; that is, the value of x ranges from 0 to 3. In fact, I can see that the value where β is maximized lies between 2 and 3. It might even be $x = 2.5$ — so P is closer to B than to A. That's what I have to say; I don't want to compute the value.

We could have used any of the inverse trigonometric functions, but I believe that $\arctan x$ would give us the simplest case; I initially tried $\arcsin x$ and I found very complicated derivatives for us to manually look for roots, so I tried $\arctan x$ instead and it became simple enough — though I didn't want to find its roots anyway.

4.8 The hyperbolic functions

The hyperbolic functions have nothing to do with trigonometric functions, but they have very similar properties. Let begin with their definitions. Define

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

as the hyperbolic sine function. Define

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

as the hyperbolic cosine. Now, like the trigonometric functions, we define $\operatorname{csch} x = 1/\sinh x$, $\operatorname{sech} x = 1/\cosh x$, $\tanh x = \sinh x/\cosh x$, and finally $\coth x = \cosh x/\sinh x$.

Notice that $\cosh x$ is the average of the functions e^x and e^{-x}, and $\sinh x$ is half of the difference between them. These functions are pretty popular in scientic contexts; they are used to describe some important phenomena. For instance, the effect of gravity on a wire hanging on posts — such as eletricity posts — on the streets is described by

$$y = c + d \cosh \frac{x}{d},$$

where c and d are constants. By the way, when a wire hangs between two posts, the the system satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{pq}{T}\sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

where p is the linear density of the wire, g is the acceleration due to gravity and T is the tension in the cable, at its lowest point. A solution to this equation is the function

$$y = \frac{T}{pg} \cosh \frac{pgx}{T}.$$

I think that linear density is a term used to describe the density of objects that have a shape that we may consider one dimensional. No objects that we know in the world are one dimensional, of course, but a wire can be considered one dimensional, so the density of a wire is a measure of mass per unit length, and we call it linear density.

These hyperbolic functions satisfy some identies which look very much like the trigonometric functions — though not always they are too similar. For instance, while $\cos^2 x + \sin^2 x = 1$, we have $\cosh^2 x - \sinh^2 x = 1$.

Let us now define the inverse of the $\sinh x$ function, and then find its derivative. To define it, let $y = \operatorname{arcsinh} x$ so that $\sinh y = x$. Then

$$\begin{aligned} x &= \sinh y \\ &= \frac{e^y - e^{-y}}{2} \\ 2x &= e^y - e^{-y} \\ e^y - e^{-y} - 2x &= 0. \end{aligned}$$

Multiply this equation by e^y to get

$$\begin{aligned} (e^y)^2 - e^y e^{-y} - 2xe^y &= 0 \\ (e^y)^2 - 2xe^y - 1 &= 0. \end{aligned}$$

Now let $p = e^y$, and notice that we have a quadratic equation, whose solution is $p = x \pm \sqrt{x^2+1}$. But since $e^y > 0$ for all y, and since $x < \sqrt{x^2+1}$ for all x, we need not consider the solution $p = x - \sqrt{x^2+1}$. Therefore

$$\begin{aligned} e^y &= x + \sqrt{x^2+1} \\ y &= \ln(x + \sqrt{x^2+1}) \\ &= \operatorname{arcsinh} x. \end{aligned}$$

The inverse hyperbolic cosine can be defined in a similar manner. Their derivatives are easily obtained by using the chain rule.

4.9 The theorem of l'Hospital

L'Hospital's theorem is a pretty interesting one. Before we state it, let us consider something. Suppose that f' and g' are continuous everywhere. Then let us consider the expression

$$\lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

Since they are both continuous, we use the theorem of limits of continuous function to write that

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = \frac{f'(c)}{g'(c)}.$$

Now, let us write the equation in terms of the definition of derivatives, and algebraically manipulate it. We write

$$\begin{aligned} \lim_{x\to c} \frac{f'(x)}{g'(x)} &= \frac{f'(c)}{g'(c)} \\ &= \frac{\lim_{x\to c} \dfrac{f(x)-f(c)}{x-c}}{\lim_{x\to c} \dfrac{g(x)-g(c)}{x-c}} \\ &= \lim_{x\to c} \frac{\dfrac{f(x)-f(c)}{x-c}}{\dfrac{g(x)-g(c)}{x-c}} \\ &= \lim_{x\to c} \frac{f(x)-f(c)}{g(x)-g(c)}. \end{aligned}$$

Now, suppose that $f(c) = g(c) = 0$. Then

$$\lim_{x\to c} \frac{f(x)-f(c)}{g(x)-g(c)} = \lim_{x\to c} \frac{f(x)}{g(x)}.$$

Let us look back now. We supposed f' and g' are continuous, and that $f(c) = g(c) = 0$; this last supposition means that $f' \to 0$ as $x \to c$; similarly for g'; that is, $g' \to 0$ as $x \to c$. And, we concluded that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

In other words, the limit of the ratio of two functions is equal to the limit of the ratio of their derivatives, provided that the derivatives are continuous over the interval where c is, and provided that the derivatives approach zero as x approaches c.

This is sort of what l'Hospital's theorem is about. We have proved a particular case of l'Hospital's theorem. The theorem is more general than this, and it applies to the case when the derivatives approach infinity too. It is harder to prove l'Hospital's theorem more generally, and I'm going to leave it at this for now.

Using algebra, the theorem of l'Hospital can be used to evaluate limits of other indeterminated forms such as $0 \cdot \infty, \infty - \infty, 0^0$, ∞^0, and 1^∞. The products, subtractions and divisions can be treated by plain algebra, and the powers can be treated by logarithms.

Contents