Chapter 2

Math 2890-001
Spring 2018 **Name** ______________________
Due Feb 28

Problems 1 – 13 are fair game for Exam 1.

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1. (1 point) Let $A = \begin{pmatrix} 2 & 1 & 9 & 4 \\ 8 & 3 & 0 & 5 \\ 3 & 5 & 5 & 4 \end{pmatrix}$. Find A^T, the transpose of A.

   answer: The transpose $A^T = \begin{pmatrix} 2 & 8 & 3 \\ 1 & 3 & 5 \\ 9 & 0 & 5 \\ 4 & 5 & 4 \end{pmatrix}$.

2. (1 point) Let $A = \begin{pmatrix} 9 & 2 & 4 \\ 2 & 8 & 9 \\ 4 & 1 & 3 \end{pmatrix}$. Is A symmetric?

   answer: A is not symmetric since $A_{23} = 9 \neq 1 = A_{32}$.

3. (1 point) Let

$$A = \begin{pmatrix} 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \end{pmatrix}.$$

Is A orthogonal? Explain your answer.

answer: A is not orthogonal since $I \neq A^T A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$.

4. (3 points) For each of the following matrices, find the inverse if it exists. Show your work.

(a) $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

answer: $A^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$.

(b) $B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

answer: $A^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$.

(c) $C = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

answer: $A^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 1 \end{pmatrix}$.

5. (1 point) Let $A = \begin{pmatrix} -2 & -1 \\ 8 & 8 \\ 0 & 1 \\ -1 & -8 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -2 \\ 6 & 5 \\ -6 & 4 \\ 8 & -3 \end{pmatrix}$.

Compute the sum $A + B$ (showing your work) if it is defined; otherwise, explain why it is not defined.

answer: The sum $A + B = \begin{pmatrix} 1 & -3 \\ 14 & 13 \\ -6 & 5 \\ 7 & -11 \end{pmatrix}$.

6. (1 point) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $v = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

Compute the product Av (showing your work) if it is defined; otherwise, explain why it is not defined.

answer: The product $Av = \begin{pmatrix} 10 \\ 4 \\ 1 \\ 6 \end{pmatrix}$.

7. (1 point) Let $v = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

Compute the product vA (showing your work) if it is defined; otherwise, explain why it is not defined.

answer: The product $vA = \begin{pmatrix} 5 & 7 & 14 \end{pmatrix}$.

8. (1 point) Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 1 & 3 \\ 1 & 2 & 3 & 1 \end{pmatrix}$.

Compute the product AB (showing your work) if it is defined; otherwise, explain why it is not defined.

answer: The product $AB = \begin{pmatrix} 4 & 5 & 7 & 5 \\ 7 & 5 & 6 & 10 \\ 6 & 6 & 8 & 8 \end{pmatrix}$.

9. (1 point) Suppose A is a 52×19 matrix and B is a 19×23 matrix. What size is the product AB if it is defined? Explain your answer.

10. (1 point) Let $A = \begin{pmatrix} 4 & 2 \\ 2 & 8 \\ 3 & 1 \\ 9 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 9 & 4 & 6 \\ 6 & 1 & 3 & 2 \end{pmatrix}$.

Compute column 3 of AB by first writing it as a linear combination of the columns of A. Show your work.

answer:

$$\begin{aligned}
(AB)_{*3} &= A(B_{*3}) \\
&= \begin{pmatrix} 4 & 2 \\ 2 & 8 \\ 3 & 1 \\ 9 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} 4 \\ 2 \\ 3 \\ 9 \end{pmatrix} (4) + \begin{pmatrix} 2 \\ 8 \\ 1 \\ 1 \end{pmatrix} (3) \\
&= \begin{pmatrix} 22 \\ 32 \\ 15 \\ 39 \end{pmatrix}
\end{aligned}$$

11. (1 point) Let $A = \begin{pmatrix} 4 & 2 \\ 2 & 8 \\ 3 & 1 \\ 9 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 9 & 4 & 6 \\ 6 & 1 & 3 & 2 \end{pmatrix}$.

Compute row 2 of AB by first writing it as a linear combination of the rows of B. Show your work.

answer:

$$\begin{aligned} (AB)_{2*} &= (A_{2*})B \\ &= \begin{pmatrix} 2 & 8 \end{pmatrix} \begin{pmatrix} 3 & 9 & 4 & 6 \\ 6 & 1 & 3 & 2 \end{pmatrix} \\ &= (2)\begin{pmatrix} 3 & 9 & 4 & 6 \end{pmatrix} + (8)\begin{pmatrix} 6 & 1 & 3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 54 & 26 & 32 & 28 \end{pmatrix} \end{aligned}$$

12. (1 point) Let $A = \begin{pmatrix} 4 & 2 \\ 2 & 8 \\ 3 & 1 \\ 9 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 9 & 4 & 6 \\ 6 & 1 & 3 & 2 \end{pmatrix}$.

Compute the entry in row 3, column 2 of the product AB. Show your work.

answer:

$$\begin{aligned} (AB)_{32} &= A_{3*}B_{*2} \\ &= \begin{pmatrix} 3 & 1 \end{pmatrix} \begin{pmatrix} 9 \\ 1 \end{pmatrix} \\ &= 28. \end{aligned}$$

13. (1 point) Let $A = \begin{pmatrix} 4 & 2 \\ 2 & 8 \\ 3 & 1 \\ 9 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 9 & 4 & 6 \\ 6 & 1 & 3 & 2 \end{pmatrix}$.

Write the product AB as a sum of 2 rank one matrices. Show your work.

answer:

$$
\begin{aligned}
AB &= A_{*1}B_{1*} + A_{*2}B_{2*} \\
&= \begin{pmatrix} 4 \\ 2 \\ 3 \\ 9 \end{pmatrix} \begin{pmatrix} 3 & 9 & 4 & 6 \end{pmatrix} + \begin{pmatrix} 2 \\ 8 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 6 & 1 & 3 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 12 & 36 & 16 & 24 \\ 6 & 18 & 8 & 12 \\ 9 & 27 & 12 & 18 \\ 17 & 81 & 36 & 54 \end{pmatrix} + \begin{pmatrix} 12 & 2 & 6 & 4 \\ 48 & 8 & 24 & 16 \\ 6 & 1 & 3 & 2 \\ 6 & 1 & 3 & 2 \end{pmatrix}
\end{aligned}
$$

The remaining problems will not be covered on Exam 1.

14. (1 point) Let

$$A = \begin{pmatrix} 5 & -4 & -1 \\ -10 & 4 & 2 \\ 25 & -8 & -3 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -3 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 5 & -4 & -1 \\ 0 & -4 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } \quad b = \begin{pmatrix} -10 \\ -8 \\ 48 \end{pmatrix}.$$

Use the LU factorization $A = LU$ to solve the matrix equation $Ax = b$. Show your work.

answer: I row reduced the augmented matrix $[L \,|\, b]$ to find that the vector $y = \begin{pmatrix} -10 \\ -28 \\ 14 \end{pmatrix}$ is the solution of the system $Ly = b$.

Then I row reduced the augmented matrix $[U \,|\, y]$ to find that the vector $x = \begin{pmatrix} 5 \\ 7 \\ 7 \end{pmatrix}$ is the solution of $Ux = y$ and so of $Ax = b$.

15. (1 point) Let

$$A = \begin{pmatrix} 4 & -2 & 5 \\ 12 & -3 & 18 \\ -4 & -13 & -17 \end{pmatrix}.$$

Use Wedderburn rank reduction (or Gaussian Elimination) to find the LDU (or LU) factorization of the matrix A. Show your work.

answer:

$$\begin{pmatrix} \mathbf{4} & \mathbf{-2} & \mathbf{5} \\ \mathbf{12} & -3 & 18 \\ \mathbf{-4} & -13 & -17 \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \\ -4 \end{pmatrix} (1/4) \begin{pmatrix} 4 & -2 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & -15 & -12 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{3} & \mathbf{3} \\ 0 & \mathbf{-15} & -12 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -15 \end{pmatrix} (1/3) \begin{pmatrix} 0 & 3 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \mathbf{0} \\ 0 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} (1/3) \begin{pmatrix} 0 & 0 & 3 \end{pmatrix}$$

So we have $A = LDU$ where

$$L = \begin{pmatrix} 4 & 0 & 0 \\ 12 & 3 & 0 \\ -4 & -15 & 3 \end{pmatrix} \quad D = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} \quad U = \begin{pmatrix} 4 & -2 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 3 \end{pmatrix}.$$

Alternatively we have $A = LU$ where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & -5 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 4 & -2 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 3 \end{pmatrix}.$$

16. (1 point) Let

$$A = \begin{pmatrix} 15 & 0 & 13 \\ -20 & 8 & -4 \\ 10 & -13 & -1 \end{pmatrix}.$$

Use Wedderburn rank reductin (or Gaussian Elimination with Partial Pivoting) to find the permuted LDU (or permuted LU) factorization of the matrix A. Show your work.

answer:

$$\begin{pmatrix} \mathbf{15} & 0 & 13 \\ \mathbf{-20} & \mathbf{8} & \mathbf{-4} \\ \mathbf{10} & -13 & -1 \end{pmatrix} = \begin{pmatrix} 15 \\ -20 \\ 10 \end{pmatrix} (-1/20) \begin{pmatrix} -20 & 8 & -4 \end{pmatrix} + \begin{pmatrix} 0 & 6 & 10 \\ 0 & 0 & 0 \\ 0 & -9 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \mathbf{6} & 10 \\ 0 & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{-9} & \mathbf{-3} \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ -9 \end{pmatrix} (-1/9) \begin{pmatrix} 0 & -9 & -3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{8} \\ 0 & 0 & \mathbf{0} \\ 0 & 0 & \mathbf{0} \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 0 \end{pmatrix} (1/8) \begin{pmatrix} 0 & 0 & 8 \end{pmatrix}$$

So we have $A = LDU$ where Either $A = LDU$ where

$$L = \begin{pmatrix} 15 & 6 & 8 \\ -20 & 0 & 0 \\ 10 & -9 & 0 \end{pmatrix} \quad D = \begin{pmatrix} -1/20 & 0 & 0 \\ 0 & -1/9 & 0 \\ 0 & 0 & 1/8 \end{pmatrix} \quad U = \begin{pmatrix} -20 & 8 & -4 \\ 0 & -9 & -3 \\ 0 & 0 & 8 \end{pmatrix},$$

Alternatively we have $A = LU$ where

$$L = \begin{pmatrix} -0.75 & -0.6667 & 1 \\ 1 & 0 & 0 \\ -0.5 & 1 & 0 \end{pmatrix} \quad U = \begin{pmatrix} -20 & 8 & -4 \\ 0 & -9 & -3 \\ 0 & 0 & 8 \end{pmatrix},$$

17. (2 points) Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

(a) Is v in the column space of A? Show your work.

(b) Is v in the null space of A? Show your work.

answer: (a) Since $(A|v)$ does **not** have a pivot in the last column $Ax = v$ is consistent, and v **is** in the column space of A.

(b) v **is** in the null space of A since $Av = 0$.

18. (2 points) Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

(a) Find a nonzero vector in the column space of A. Show your work.

(b) Find a nonzero vector in the null space of A. Show your work.

answer: (a) I chose the vector $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, a column of A so obviously in the column space. You'll probably find a different vector.

(b) I row reduced the augmented matrix $[A \,|\, 0]$ to find the vector $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ in the null space. You may find a different vector.

19. (1 point) Let

$$A = \begin{pmatrix} -5 & 5 & 2 & 25 & 18 & -2 \\ -2 & 0 & 2 & 10 & 2 & -1 \\ -2 & -4 & 4 & 8 & -8 & -4 \\ -5 & -3 & 5 & 16 & -1 & 0 \\ 2 & 2 & 0 & 6 & 0 & -2 \end{pmatrix}.$$

Find bases for Col(A) and Nul(A).

Hint: The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & 5 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

answer:

Col(A): A basis for the column space of A is provided by the columns of A that end up with pivots in the echelon form matrix R (columns 1, 2, 3, and 6):

$$\begin{pmatrix} -5 \\ -2 \\ -2 \\ -5 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ -4 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 4 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ -4 \\ 0 \\ -2 \end{pmatrix}.$$

Nul(A): We need to find the solution of the homogeneous equation $Ax = 0$. Augmenting R with the 0 vector and observing that $x_4 = r$ and $x_5 = s$ are free variables while $x_1 = 2s$, $x_2 = -3r - 2s$, $x_3 = -5r + s$, and $x_6 = 0$, so

$$x_H = \begin{pmatrix} 0 \\ -3 \\ -5 \\ 1 \\ 0 \\ 0 \end{pmatrix} r + \begin{pmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} s\text{; and we have the basis } \begin{pmatrix} 0 \\ -3 \\ -5 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

20. (1 point) Let

$$A = \begin{pmatrix} 58 & -15 & -144 & -204 & 11 \\ 101 & -27 & -249 & -357 & -4 \\ -45 & 12 & 111 & 159 & 1 \\ 15 & -4 & -37 & -53 & 0 \end{pmatrix}.$$

Find bases for Col(A), Nul(A), Col(A^T) and Nul(A^T).

Hint: If you form the matrix $(A|I)$ and use row operations to put the A part in reduced row echelon form you get $(R|S)$ where

$$R = \begin{pmatrix} 1 & 0 & -3 & -3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ S = \begin{pmatrix} 1 & 3 & 1 & -21 \\ 4 & 13 & 8 & -79 \\ -1 & -7 & -16 & 3 \\ 1 & 7 & 17 & 0 \end{pmatrix}.$$

answer:

Col(A): A basis for the column space of A is provided by the columns of A that end up with pivots in the echelon form matrix R (columns 1, 2, and 5):

$$\begin{pmatrix} 58 \\ 101 \\ -45 \\ 15 \end{pmatrix}, \begin{pmatrix} -15 \\ -27 \\ 12 \\ -4 \end{pmatrix}, \begin{pmatrix} 11 \\ -4 \\ 1 \\ 0 \end{pmatrix}.$$

Nul(A): We need to find the solution of the homogeneous equation $Ax = 0$. Augmenting R with the 0 vector and observing that $x_3 = r$ and $x_4 = s$ are free variables while $x_1 = 3r + 3s$, $x_2 = 2r - 2s$, and $x_5 = 0$, so

$$x_H = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} r + \begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} s\text{; and we have the basis } \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Col(A^T): A basis for the column space of A^T is provided by the transposes of the nonzero rows of R:

$$\begin{pmatrix} 1 \\ 0 \\ -3 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

$\mathsf{Nul}(A^T)$: A basis for the null space of A^T is provided by the transposes of the rows of S that sit next to the rows of R that have nothing but zeros in them:

$$\begin{pmatrix} 1 \\ 7 \\ 17 \\ 0 \end{pmatrix}.$$

21. (1 point) Let

$$A = \begin{pmatrix} 1 & 2 & 8 & 1 & 5 \\ 2 & 1 & 7 & 2 & 7 \\ 1 & 1 & 5 & 0 & 2 \\ 3 & 2 & 12 & 1 & 7 \end{pmatrix}.$$

Find the rank of the matrix A. Explain your answer.

answer: The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

a matrix having 3 pivots, so the rank of A is 3.