Chapter 6

Math 2890-003
Spring 2016 **Name** ______________________
Due Mar 17

1. (1 point) Let

$$u = \begin{pmatrix} -1 \\ 1 \\ -7 \\ 8 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 4 \\ -6 \\ 7 \\ -7 \end{pmatrix}.$$

Find the inner product $u \cdot v$. Show your work.

answer: $u \cdot v = \sum_k u_k v_k = (-1)(4) + (1)(-6) + (-7)(7) + (8)(-7) = -115.$

2. (1 point) Let

$$v = \begin{pmatrix} 3 \\ -5 \\ -3 \\ 7 \end{pmatrix}.$$

Find a unit vector in the direction of v. Show your work.

answer: Since $\|v\|^2 = v \cdot v = (3)^2 + (-5)^2 + (-3)^2 + (7)^2 = 92$, the unit vector

$$u = \pm \frac{v}{\|v\|} = \pm \begin{pmatrix} 3 \\ -5 \\ -3 \\ 7 \end{pmatrix} \frac{1}{\sqrt{92}} = \pm \begin{pmatrix} 0.3128 \\ -0.5213 \\ -0.3128 \\ 0.7298 \end{pmatrix}.$$

3. (1 point) Let

$$u = \begin{pmatrix} 4 \\ -7 \\ 6 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}.$$

Find the distance between u and v. Show and explain your computations.

answer: The distance is $\|u - v\| = \sqrt{149} = 12.2066$, where $u - v = \begin{pmatrix} 1 \\ -12 \\ 2 \end{pmatrix}$.

4. (1 point) Let

$$u_1 = \begin{pmatrix} 5 \\ -1 \\ 1 \\ -5 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 2 \\ 8 \\ 8 \\ 2 \end{pmatrix} \quad \text{and} \quad u_3 = \begin{pmatrix} 7 \\ -7 \\ 3 \\ 9 \end{pmatrix}.$$

Is the set $\{u_1, u_2, u_3\}$ orthogonal? Why or why not? Show your computations.

answer: Yes, $u_i \cdot u_j = 0$ for all $i \neq j$.

5. (1 point) Let

$$y = \begin{pmatrix} -5 \\ 2 \\ -5 \end{pmatrix}$$

and let W be the span of

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix}.$$

Project y onto W. Show and explain your computations.

answer: The projection is

$$\begin{aligned} w &= A(A^T A)^{-1}(A^T y) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 5 & -4 \\ -4 & 8 \end{pmatrix}^{-1} \begin{pmatrix} -15 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} -4 \\ -1.25 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ 2.5 \\ -5.5 \end{pmatrix} \end{aligned}$$

6. (1 point) Let

$$y = \begin{pmatrix} 1 \\ -1 \\ -7 \end{pmatrix}$$

and let W be the span of

$$\begin{pmatrix} 1 \\ -5 \\ -4 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ -8 \\ 2 \end{pmatrix}.$$

Find the point in W that is closest to y. Show and explain your computations.

answer: The closest point is

$$\begin{aligned} w &= A(A^TA)^{-1}(A^Ty) \\ &= \begin{pmatrix} 1 & 2 \\ -5 & -8 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} 42 & 34 \\ 34 & 72 \end{pmatrix}^{-1} \begin{pmatrix} 34 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ -5 & -8 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} 1.3833 \\ -0.7088 \end{pmatrix} \\ &= \begin{pmatrix} -0.0343 \\ -1.2463 \\ -6.9507 \end{pmatrix} \end{aligned}$$

7. (1 point) Let

$$y = \begin{pmatrix} 5 \\ 4 \\ 2 \\ -2 \end{pmatrix}$$

and let W be the span of

$$\begin{pmatrix} -2 \\ 2 \\ 0 \\ -2 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ -8 \\ -5 \\ -7 \end{pmatrix}.$$

Write y as a sum of a vector in W and a vector orthogonal to W. Show and explain your computations.

answer: The vector in W is the projection

$$\begin{aligned} w &= A(A^T A)^{-1}(A^T y) \\ &= \begin{pmatrix} -2 & 3 \\ 2 & -8 \\ 0 & -5 \\ -2 & -7 \end{pmatrix} \begin{pmatrix} 12 & -8 \\ -8 & 147 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ -13 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 3 \\ 2 & -8 \\ 0 & -5 \\ -2 & -7 \end{pmatrix} \begin{pmatrix} 0.1118 \\ -0.0824 \end{pmatrix} \\ &= \begin{pmatrix} -0.4706 \\ 0.8824 \\ 0.4118 \\ 0.3529 \end{pmatrix}. \end{aligned}$$

The vector orthogonal to W is

$$\begin{aligned} v &= y - w \\ &= \begin{pmatrix} 5 \\ 4 \\ 2 \\ -2 \end{pmatrix} - \begin{pmatrix} -0.4706 \\ 0.8824 \\ 0.4118 \\ 0.3529 \end{pmatrix} \\ &= \begin{pmatrix} 5.4706 \\ 3.1176 \\ 1.5882 \\ -2.3529 \end{pmatrix}. \end{aligned}$$

And we have

$$y = \begin{pmatrix} 5 \\ 4 \\ 2 \\ -2 \end{pmatrix} = w + v = \begin{pmatrix} -0.4706 \\ 0.8824 \\ 0.4118 \\ 0.3529 \end{pmatrix} + \begin{pmatrix} 5.4706 \\ 3.1176 \\ 1.5882 \\ -2.3529 \end{pmatrix}.$$

8. (1 point) Let

$$A = \begin{pmatrix} -1 & -2 \\ 0 & 2 \\ 2 & 6 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -4 \\ 1 \\ -5 \\ 1 \end{pmatrix}.$$

Find the least squares solution to $Ax = b$. Show and explain your computations.

answer: The normal equations $A^T Ax = A^T b$

are $\begin{pmatrix} 9 & 18 \\ 18 & 48 \end{pmatrix} x = \begin{pmatrix} -4 \\ -18 \end{pmatrix}$,

and these have solution $x = \begin{pmatrix} 1.2222 \\ -0.8333 \end{pmatrix}$.

9. (1 point) Let

$$A = \begin{pmatrix} 1 & -5 \\ 4 & -1 \\ -2 & -1 \\ 1 & -2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 3 \\ 3 \\ 2 \\ -2 \end{pmatrix}.$$

Find the least squares error in the least squares solution to $Ax = b$. Show and explain your computations.

HINT: The least squares solution is $x = \begin{pmatrix} 0.2246 \\ -0.4509 \end{pmatrix}$.

answer: The least squares error is $\|Ax - b\| = 4.0944$, since $Ax - b =$
$$\begin{pmatrix} 2.4792 \\ 1.3494 \\ 0.0017 \\ 1.1265 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -0.5208 \\ -1.6506 \\ -1.9983 \\ 3.1265 \end{pmatrix}.$$

10. (1 point) Use the QR factorizaton

$$A = \begin{pmatrix} -0.199 & 0.4302 \\ 0 & -1.5113 \\ 1.5921 & 2.6036 \\ 1.194 & 4.9752 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} -0.0995 & 0.3092 \\ 0 & -0.5038 \\ 0.796 & -0.4589 \\ 0.597 & 0.6634 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} 2 & 5 \\ 0 & 3 \end{pmatrix}}_{R}.$$

to find the least squares solution to $Ax = b$, where $b = \begin{pmatrix} -4 \\ 0 \\ -4 \\ 5 \end{pmatrix}$.

Show your work.

answer: The equation $Rx = Q^T b$ is $\begin{pmatrix} 2 & 5 \\ 0 & 3 \end{pmatrix} x = \begin{pmatrix} 0.199 \\ 3.9153 \end{pmatrix}$,

and this has solution $x = \begin{pmatrix} -3.1633 \\ 1.3051 \end{pmatrix}$.

11. (1 point) Let

$$A = \begin{pmatrix} -2 & -4 & 11 \\ -4 & -13 & 17 \\ 5 & 15 & -18 \end{pmatrix}.$$

Find the QR factorization of A. Show and explain your computations.

answer: $A = QR$ where

$$Q = \begin{pmatrix} -2/\sqrt{45} & 2/\sqrt{5} & 1/\sqrt{9} \\ -4/\sqrt{45} & -1/\sqrt{5} & 2/\sqrt{9} \\ 5/\sqrt{45} & 0 & 2/\sqrt{9} \end{pmatrix} = \begin{pmatrix} -0.2981 & 0.8944 & 0.3333 \\ -0.5963 & -0.4472 & 0.6667 \\ 0.7454 & 0 & 0.6667 \end{pmatrix}$$

and

$$R = \begin{pmatrix} 45/\sqrt{45} & 135/\sqrt{45} & -180/\sqrt{45} \\ 0 & 5/\sqrt{5} & 5/\sqrt{5} \\ 0 & 0 & 9/\sqrt{9} \end{pmatrix} = \begin{pmatrix} 6.7082 & 20.1246 & -26.8328 \\ 0 & 2.2361 & 2.2361 \\ 0 & 0 & 3 \end{pmatrix}.$$

12. (1 point) Use the QDR factorization

$$A = \begin{pmatrix} -7 & 29 & -36 \\ 3 & -11 & 6 \\ 3 & -11 & 14 \\ 2 & -9 & 7 \\ 3 & -11 & -2 \end{pmatrix} = \underbrace{\begin{pmatrix} -7 & 1 & -3 \\ 3 & 1 & -1 \\ 3 & 1 & 7 \\ 2 & -1 & -6 \\ 3 & 1 & -9 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} 1/80 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/176 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 80 & -320 & 320 \\ 0 & 5 & -25 \\ 0 & 0 & 176 \end{pmatrix}}_{R}$$

to find the least squares solution to $Ax = b$ where $b = \begin{pmatrix} -5 \\ 5 \\ -4 \\ 5 \\ -4 \end{pmatrix}$.

Show your work.

answer: Since $Q^T A = Q^T QDR = R$, we just need to solve the equation $Rx = Q^T b$.

This equation is $\begin{pmatrix} 80 & -320 & 320 \\ 0 & 5 & -25 \\ 0 & 0 & 176 \end{pmatrix} x = \begin{pmatrix} 36 \\ -13 \\ -12 \end{pmatrix}$, and has solution $x = \begin{pmatrix} -11.0409 \\ -2.9409 \\ -0.0682 \end{pmatrix}$.

13. (1 point) Let

$$A = \begin{pmatrix} -1 & 2 & -8 \\ 3 & -6 & 20 \\ -5 & 15 & -11 \\ -2 & 9 & 13 \\ 1 & -7 & -17 \end{pmatrix}.$$

Find the QDR factorization of A. Show and explain your computations.

answer:

$$Q = \begin{pmatrix} -1 & -1 & -1 \\ 3 & 3 & -1 \\ -5 & 0 & -1 \\ -2 & 3 & 2 \\ 1 & -4 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1/40 & 0 & 0 \\ 0 & 1/35 & 0 \\ 0 & 0 & 1/8 \end{pmatrix}$$

$$R = \begin{pmatrix} 40 & -120 & 80 \\ 0 & 35 & 175 \\ 0 & 0 & 8 \end{pmatrix}.$$

14. **(1 point)** Consider the data points $(1, 2), (2, -2), (3, -6), (4, 3), (5, 5)$.

Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that best fits the given data points. Show and explain your computations.

answer: I need to find the least squares solution of

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -6 \\ 3 \\ 5 \end{pmatrix}.$$

The normal equations are $\begin{pmatrix} 5 & 15 \\ 15 & 55 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 17 \end{pmatrix}$, and these have solution $\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} -2.9 \\ 1.1 \end{pmatrix}$. The least squares line is

$$y = -2.9 + 1.1x.$$

15. (1 point) Consider the data points $(1,1),(2,-6),(3,4),(4,-9),(5,2)$.

Find the equation $y = \beta_0 + \beta_1 x + \beta_2 x^2$ of the least-squares quadratic that best fits the given data points. Show and explain your computations.

answer: I need to find the least squares solution of

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ 4 \\ -9 \\ 2 \end{pmatrix}.$$

The normal equations are $\begin{pmatrix} 5 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} -8 \\ -25 \\ -81 \end{pmatrix}$, and these have solution $\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 5.2 \\ -5.6714 \\ 0.9286 \end{pmatrix}$. The least squares curve is

$$y = 5.2 - 5.6714x + 0.9286x^2.$$

Total for assignment: 15 points