Chapter 5

Math 2890-003
Spring 2016 **Name** ____________________
Due Apr 26

1. (1 point) Let

$$A = \begin{pmatrix} 0 & 48 & -38 & -8 \\ 12 & -43 & 24 & 12 \\ 14 & -48 & 25 & 14 \\ 19 & -120 & 82 & 27 \end{pmatrix} \quad \text{and} \quad \lambda = 8.$$

Find an eigenvector for the matrix A that corresponds to the given eigenvalue λ. Show and explain your work.

answer: An eigenvector corresponding to the eigenvalue $\lambda = 8$ is a nonzero solution of the equation $(A - (8)I)x = 0$, where

$$A - (8)I = \begin{pmatrix} -8 & 48 & -38 & -8 \\ 12 & -51 & 24 & 12 \\ 14 & -48 & 17 & 14 \\ 19 & -120 & 82 & 19 \end{pmatrix}$$

. After row reduction, it can be seen that one choice is $x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$.

2. (1 point) Let

$$A = \begin{pmatrix} 72 & -5 & 5 & -28 \\ -20 & -9 & -7 & -4 \\ -196 & 20 & -4 & 98 \\ 182 & -10 & 17 & -61 \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} 1 \\ 0 \\ -4 \\ 2 \end{pmatrix}.$$

Find the eigenvalue for the matrix A that corresponds to the given eigenvector x. Show and explain your work.

answer: The eigenvalue λ can be found from the equation $Ax = x\lambda$. After computing

$$Ax = \begin{pmatrix} -4 \\ 0 \\ 16 \\ -8 \end{pmatrix},$$

I can see that $\lambda = -4$.

3. (1 point) Let

$$A = \begin{pmatrix} -17 & 7 \\ -42 & 18 \end{pmatrix}.$$

Find the eigenvalues (including multiplicities) of A. Show and explain your work.

answer: The eigenvalues are the roots of the characteristic polynomial $\det(A - \lambda I)$. Observe that A is 2×2, so its characteristic polynomial is

$$\begin{aligned} \lambda^2 - \operatorname{tr}(A)\lambda + \det(A) &= \lambda^2 - \lambda - 12 \\ &= (\lambda - 4)(\lambda + 3) \end{aligned}$$

The eigenvalues of A are $4, -3$.

4. (1 point) Let

$$A = \begin{pmatrix} -1 & 6 & 18 \\ 0 & -4 & -21 \\ 0 & 0 & 3 \end{pmatrix}.$$

Find the eigenvalues (including multiplicities) of A. Show and explain your work.

answer: The eigenvalues are the roots of the characteristic polynomial $\det(A - \lambda I)$. Observe that A (and so $A - \lambda I$) is block triangular. Since the determinant of a triangular matrix is the product of the diagonal entries, the eigenvalues of A are the diagonal entries of $A:\ -1, -4, 3$.

5. (1 point) Let

$$A = \begin{pmatrix} 13 & -9 & 0 & 0 \\ 18 & -14 & 0 & 0 \\ 10 & -6 & -8 & -2 \\ -2 & 6 & 12 & 2 \end{pmatrix}.$$

Find the eigenvalues (including multiplicities) of A. Show and explain your work.

answer: The eigenvalues are the roots of the characteristic polynomial $\det(A - \lambda I)$. Observe that A (and so $A - \lambda I$) is block triangular. Since the determinant of a block triangular matrix is the product of the determinants of the diagonal blocks, the eigenvalues of A are the eigenvalues of the diagonal blocks of A : $4, -5, -2, -4$.

6. (1 point) Let

$$A = \begin{pmatrix} -3 & 0 & 0 \\ 6 & -5 & 0 \\ -12 & 4 & -1 \end{pmatrix}.$$

Find an invertible matrix P and a diagonal matrix D such that $AP = PD$, or explain why no such matrices exist.

answer: One choice is $P = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

7. (1 point) Let

$$A = \begin{pmatrix} 5 & 3 & 6 & -24 \\ 0 & 4 & 0 & -18 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

Find an invertible matrix P and a diagonal matrix D such that $AP = PD$, or explain why no such matrices exist.

answer: One choice is $P = \begin{pmatrix} 1 & -3 & 3 & 3 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$.

8. (1 point) Let

$$A = \begin{pmatrix} -3 & 5 & -3 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

Find an invertible matrix P and a diagonal matrix D such that $AP = PD$, or explain why no such matrices exist.

answer: No such matrices exits because -3 is an eigenvalue whose multiplicity is greater than the dimension of its eigenspace.

9. (1 point) Let

$$A = \begin{pmatrix} 1 & 1 & 19 & 9 \\ -2 & -2 & -31 & -13 \\ 0 & 0 & -7 & -2 \\ 0 & 0 & 12 & 3 \end{pmatrix}.$$

Find an invertible matrix P and a diagonal matrix D such that $AP = PD$, or explain why no such matrices exist.

answer: One choice is $P = \begin{pmatrix} 1 & 1 & -1 & -3 \\ -2 & -1 & 3 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

10. (1 point) Let $A = PDP^{-1}$ where

$$P = \begin{pmatrix} 1 & 3 & -1 & 0 \\ -2 & -5 & -2 & 1 \\ 2 & 9 & -13 & 1 \\ 4 & 13 & -7 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}.$$

Then

$$P^{-1} = \begin{pmatrix} 45 & -8 & 8 & -19 \\ -16 & 3 & -3 & 7 \\ -4 & 1 & -1 & 2 \\ 2 & 2 & -1 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} -91 & 17 & -17 & 40 \\ 178 & -21 & 27 & -69 \\ -226 & 51 & -45 & 105 \\ -388 & 71 & -71 & 169 \end{pmatrix}.$$

Find the eigenvalues of A and for each eigenvalue find a corresponding eigenvector.

answer: The factorization $A = PDP^{-1}$ where D is diagonal shows us that the eigenvalues of A are the diagonal entries of D, and that the corresponding columns of P are the desired eigenvectors. That is, we have eigenvalues

$$\lambda_1 = 1 \quad \lambda_2 = 3 \quad \lambda_3 = 2 \quad \lambda_4 = 6$$

with corresponding eigenvectors

$$x_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \\ 4 \end{pmatrix} \quad x_2 = \begin{pmatrix} 3 \\ -5 \\ 9 \\ 13 \end{pmatrix} \quad x_3 = \begin{pmatrix} -1 \\ -2 \\ -13 \\ -7 \end{pmatrix} \quad x_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

11. (1 point) Let

$$x_1 = \begin{pmatrix} 9 \\ -2 \\ 5 \\ -1 \end{pmatrix} \quad x_2 = \begin{pmatrix} -4 \\ -6 \\ 1 \\ -5 \end{pmatrix} \quad x_3 = \begin{pmatrix} -6 \\ -1 \\ 8 \\ 7 \end{pmatrix} \quad x_4 = \begin{pmatrix} -1 \\ 8 \\ 0 \\ 3 \end{pmatrix}$$

and

$$\lambda_1 = -5,\ \lambda_2 = 1,\ \lambda_3 = -8,\ \lambda_4 = 7.$$

Write down a matrix A that has the given vectors as eigenvectors with the corresponding scalars as the eigenvalues.

answer: One solution is $A = PDP^{-1}$ where $P = \begin{pmatrix} 9 & -4 & -6 & -1 \\ -2 & -6 & -1 & 8 \\ 5 & 1 & 8 & 0 \\ -1 & -5 & 7 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} -5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$.

12. (1 point) Let

$$A = \begin{pmatrix} 0.2 & -10.8 & 2.7 \\ 1.1 & 13.7 & -3.1 \\ 4.4 & 50.4 & -11.3 \end{pmatrix}.$$

Is the origin an attractor, repeller or saddle point for the discrete dynamical system $x_{k+1} = Ax_k$? How do you know?

HINT: It may help to know that $AP = PD$ where

$$P = \begin{pmatrix} 1 & 0 & 3 \\ -1 & 1 & -1 \\ -4 & 4 & -3 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 1.3 & 0 \\ 0 & 0 & 1.1 \end{pmatrix}.$$

answer: The origin is a saddle point since A has eigenvalues both greater than and less than 1 in absolute value.

13. (1 point) Let

$$A = \begin{pmatrix} -56.1 & -1.6 & -13.6 & -6 \\ -115.2 & -2.5 & -27.6 & -12 \\ 233.1 & 8.8 & 57.7 & 24.3 \\ 53.1 & -3.2 & 10.2 & 7 \end{pmatrix}.$$

Is the origin an attractor, repeller or saddle point for the discrete dynamical system $x_{k+1} = Ax_k$? How do you know?

HINT: It may help to know that $AP = PD$ where

$$P = \begin{pmatrix} 1 & 4 & -4 & -4 \\ 2 & 9 & -12 & -9 \\ -4 & -18 & 25 & 19 \\ -1 & 0 & -15 & -2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1.1 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0 & 1.6 & 0 \\ 0 & 0 & 0 & 1.9 \end{pmatrix}.$$

answer: The origin is a repeller since all eigenvalues of A are greater than 1 in absolute value.

14. (1 point) Let

$$A = \begin{pmatrix} 12.3 & 2.4 & -1.1 \\ -30.8 & -5.8 & 2.8 \\ 59.4 & 12 & -5.2 \end{pmatrix}.$$

Is the origin an attractor, repeller or saddle point for the discrete dynamical system $x_{k+1} = Ax_k$? How do you know?

HINT: It may help to know that $AP = PD$ where

$$P = \begin{pmatrix} 1 & 2 & 1 \\ -4 & -7 & 0 \\ 2 & 6 & 11 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}.$$

answer: The origin is an attractor since all eigenvalues of A are less than 1 in absolute value.

15. (1 point) Let

$$A = \begin{pmatrix} 28 & -6 & -42 \\ -22 & 4 & 34 \\ 26 & -6 & -40 \end{pmatrix}.$$

Is the origin an attractor, repeller or saddle point for the differential equation $y' = Ay$? How do you know?

HINT: It may help to know that $AP = PD$ where

$$P = \begin{pmatrix} 1 & -1 & -3 \\ -1 & 2 & 1 \\ 1 & -1 & -2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -8 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

answer: The origin is a saddle point since A has both positive and negative eigenvalues.

16. (1 point) Let

$$A = \begin{pmatrix} -47 & 24 & 9 & -3 \\ 69 & -51 & -16 & 5 \\ -393 & 244 & 83 & -29 \\ -81 & 52 & 19 & -13 \end{pmatrix}.$$

Is the origin an attractor, repeller or saddle point for the differential equation $y' = Ay$? How do you know?

HINT: It may help to know that $AP = PD$ where

$$P = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & -2 & -1 \\ -2 & -7 & 11 & 4 \\ -3 & -7 & 3 & 4 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -8 & 0 & 0 & 0 \\ 0 & -8 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -7 \end{pmatrix}.$$

answer: The origin is an attractor since all eigenvalues of A are negative.

17. (1 point) Let

$$A = \begin{pmatrix} 122 & -90 & -51 \\ 303 & -229 & -133 \\ -258 & 200 & 119 \end{pmatrix}.$$

Is the origin an attractor, repeller or saddle point for the differential equation $y' = Ay$? How do you know?

HINT: It may help to know that $AP = PD$ where

$$P = \begin{pmatrix} 1 & -3 & -3 \\ 3 & -8 & -5 \\ -3 & 7 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

answer: The origin is a repeller since all eigenvalues of A are positive.

18. (1 point) Suppose $AP = PD$ where

$$P = \begin{pmatrix} 1 & -1 & -3 & 1 \\ 4 & 1 & 2 & 0 \\ 0 & 5 & 0 & -3 \\ 5 & -2 & -1 & -4 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -9 & 0 & 0 & 0 \\ 0 & -8 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Solve the discrete dynamical system $x_{k+1} = Ax_k$ where

$$x_0 = \begin{pmatrix} 8 \\ -19 \\ 10 \\ -46 \end{pmatrix}.$$

Show your work.

answer: $x_k = \begin{pmatrix} 1 \\ 4 \\ 0 \\ 5 \end{pmatrix} (-4)(-9)^k + \begin{pmatrix} -1 \\ 1 \\ 5 \\ -2 \end{pmatrix} (5)(-8)^k + \begin{pmatrix} -3 \\ 2 \\ 0 \\ -1 \end{pmatrix} (-4)(-5)^k +$
$\begin{pmatrix} 1 \\ 0 \\ -3 \\ -4 \end{pmatrix} (5)(3)^k$

19. (1 point) Suppose $AP = PD$ where

$$P = \begin{pmatrix} 1 & -1 & -2 & 0 \\ 4 & 3 & -2 & 0 \\ 5 & -5 & 5 & -2 \\ -2 & 2 & -5 & -4 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Solve the initial value problem $y' = Ay$ where

$$y(0) = \begin{pmatrix} -4 \\ -32 \\ 12 \\ -6 \end{pmatrix}.$$

Show your work.

answer: $y = \begin{pmatrix} 1 \\ 4 \\ 5 \\ -2 \end{pmatrix} (-4)e^{-4t} + \begin{pmatrix} -1 \\ 3 \\ -5 \\ 2 \end{pmatrix} (-4)e^{4t} + \begin{pmatrix} -2 \\ -2 \\ 5 \\ -5 \end{pmatrix} (2)e^{8t} +$
$\begin{pmatrix} 0 \\ 0 \\ -2 \\ -4 \end{pmatrix} (-1)e^{t}$

20. (1 point) Let

$$A = \begin{pmatrix} 7 & 24 & 16 \\ -16 & -43 & -26 \\ 16 & 39 & 22 \end{pmatrix} \quad \text{and} \quad x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Use the Power Method to find estimates μ_5 and x_5 for the dominant eigenvalue of A and its eigenvector. Give your answer either as rational numbers or decimals with at least four digits of accuracy.

answer: $\mu_5 = -9.0292$ and $x_5 = \begin{pmatrix} -0.5029 \\ 1 \\ -0.9971 \end{pmatrix}$

details: The vector $x_{k+1} = Ax_k(1/\mu_k)$ where μ_k is an entry in Ax_k whose absolute value is as large as possible.

$$x_k = \begin{pmatrix} 1 & -0.5529 & -0.5316 & -0.5147 & -0.5065 & -0.5029 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -0.9059 & -0.9645 & -0.9849 & -0.9934 & -0.9971 \end{pmatrix}$$

$$\mu_k = \begin{pmatrix} -85 & -10.6 & -9.4173 & -9.1575 & -9.0666 & -9.0292 \end{pmatrix}$$

21. (1 point) Let

$$A = \begin{pmatrix} 3.7 & 5.2 & 3.6 \\ -31.2 & 12.1 & -10.4 \\ -21.6 & -10.4 & -14.3 \end{pmatrix}, \; x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } \alpha = 2.$$

Use the Inverse Power Method to find the estimate ν_3 for the eigenvalue of A closest to α and the estimate x_3 for the corresponding eigenvector. Give your answer as decimals with at least four digits of accuracy.

answer: $\nu_3 = 1.7000$ and $x_3 = \begin{pmatrix} -0.5 \\ -0.5001 \\ 1 \end{pmatrix}$.

details: The vector $x_{k+1} = y_k/\mu_k$ where y_k is a solution of $(A-\alpha I)y_k = x_k$ and μ_k is an entry in y_k whose absolute value is as large as possible. The eigenvalue estimate $\nu_k = \alpha + 1/\mu_k$.

$$x_k = \begin{pmatrix} 1 & -0.5035 & -0.5001 & -0.5 \\ 1 & -0.5236 & -0.4995 & -0.5001 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$
$$\mu_k = \begin{pmatrix} 46.9021 & -3.309 & -3.3413 & -3.3331 \end{pmatrix}$$
$$\nu_k = \begin{pmatrix} 2.0213 & 1.6978 & 1.7007 & 1.7 \end{pmatrix}$$

Total for assignment: 21 points