Math 2890-003
Spring – 2016
Midterm 1
Odenthal

> **Instructions:** No books. No notes. Non-graphing calculators only. Please write neatly. There are 14 problems on 10 pages worth 250 points. **Show your work! Explain your answers.**

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1. (20 points) Let $A = \begin{pmatrix} -4 & -3 & -10 & 5 \\ 4 & 2 & 8 & 2 \\ 3 & 3 & 9 & -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 36 \\ 18 \end{pmatrix}$.

   Find the general solution of the equation $A\mathbf{x} = \mathbf{b}$. Show your work.

   HINT: The reduced row echelon form of the augmented matrix $(A \,|\, \mathbf{b})$ is

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right).$$

   answer: The general solution is

$$\mathbf{x} = \begin{pmatrix} 2 \\ 8 \\ 0 \\ 6 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} r$$

   where $r \in \mathbb{R}$ is arbitrary.

2. (15 points) Let $\alpha = 5$, $\beta = 3$, $\mathbf{v} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$.

Compute the linear combination $\mathbf{v}\alpha + \mathbf{w}\beta$, or explain why it is impossible. Show your work.

answer: The linear combination

$$\begin{aligned} \mathbf{v}\alpha + \mathbf{w}\beta &= \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix}(5) + \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}(3) \\ &= \begin{pmatrix} 20 \\ 25 \\ 10 \end{pmatrix} + \begin{pmatrix} 0 \\ -18 \\ 9 \end{pmatrix} \\ &= \begin{pmatrix} 20 \\ 7 \\ 19 \end{pmatrix} \end{aligned}$$

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3. (15 points) Let $A = \begin{pmatrix} -5 & -5 \\ 3 & 8 \\ -4 & -7 \\ 7 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 8 & -2 & -5 & 8 \\ 5 & -4 & 8 & 7 & 2 \end{pmatrix}$.

Write column 4 of the matrix product AB as a linear combination of the columns of A.

answer:

$$\begin{aligned} (AB)_{*4} &= A(B_{*4}) \\ &= \begin{pmatrix} -5 & -5 \\ 3 & 8 \\ -4 & -7 \\ 7 & 7 \end{pmatrix} \begin{pmatrix} -5 \\ 7 \end{pmatrix} \\ &= \begin{pmatrix} -5 \\ 3 \\ -4 \\ 7 \end{pmatrix}(-5) + \begin{pmatrix} -5 \\ 8 \\ -7 \\ 7 \end{pmatrix}(7) \end{aligned}$$

4. (15 points) Let $A = \begin{pmatrix} -4 & -9 & -3 \\ 2 & 9 & 3 \\ -1 & 4 & 6 \\ 9 & 8 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & 1 & -5 & 3 \\ -6 & -5 & -3 & -3 \\ 1 & 7 & -1 & -4 \end{pmatrix}$.

Compute the entry of AB in row 3, column 4.

answer:

$$\begin{aligned} (AB)_{34} &= A_{3*}B_{*4} \\ &= \begin{pmatrix} -1 & 4 & 6 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \\ -4 \end{pmatrix} \\ &= (-1)(3) + (4)(-3) + (6)(-4) \\ &= -39 \end{aligned}$$

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5. (15 points) Let $A = \begin{pmatrix} 2 & 4 \\ 4 & -2 \\ 8 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 9 & -2 & 9 & -1 \\ -5 & 3 & 2 & 9 \end{pmatrix}$.

Write the product AB as a sum of 2 rank one matrices.

answer:

$$\begin{aligned} AB &= A_{*1}B_{1*} + A_{*2}B_{2*} \\ &= \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix} \begin{pmatrix} 9 & -2 & 9 & -1 \end{pmatrix} + \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix} \begin{pmatrix} -5 & 3 & 2 & 9 \end{pmatrix} \\ &= \begin{pmatrix} 18 & -4 & 18 & -2 \\ 36 & -8 & 36 & -4 \\ 72 & -16 & 72 & -8 \end{pmatrix} + \begin{pmatrix} -20 & 12 & 8 & 36 \\ 10 & -6 & -4 & -18 \\ -20 & 12 & 8 & 36 \end{pmatrix} \end{aligned}$$

6. (20 points) Let $A = \begin{pmatrix} 3 & 1 & 5 \\ 12 & 6 & 18 \\ 6 & 8 & 9 \end{pmatrix}$.

Use Wedderburn rank reduction (or Gaussian Elimination) to find the LDU (or LU) factorization of the matrix A.

answer: $A = LDU$ where

$$L = \begin{pmatrix} 3 & 0 & 0 \\ 12 & 2 & 0 \\ 6 & 6 & 5 \end{pmatrix},$$

$$D = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/5 \end{pmatrix},$$

$$U = \begin{pmatrix} 3 & 1 & 5 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{pmatrix}.$$

Or else $A = LU$ with the same U, but with

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}.$$

7. (20 points) Let $A = \begin{pmatrix} 9 & -3 & -4 \\ 15 & -15 & -10 \\ 6 & 3 & 8 \end{pmatrix}$.

Use Wedderburn rank reduction (or Gaussian Elimination with Partial Pivoting) to find the *permuted* LDU (or *permuted* LU) factorization of the matrix A.

answer: $A = LDU$ where

$$L = \begin{pmatrix} 9 & 6 & -6 \\ 15 & 0 & 0 \\ 6 & 9 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 1/15 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & -1/6 \end{pmatrix},$$

$$U = \begin{pmatrix} 15 & -15 & -10 \\ 0 & 9 & 12 \\ 0 & 0 & -6 \end{pmatrix}.$$

Or else $A = LU$ with the same U, but with

$$L = \begin{pmatrix} 3/5 & 2/3 & 1 \\ 1 & 0 & 0 \\ 2/5 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.6667 & 1.0 \\ 1.0 & 0.0 & 0.0 \\ 0.4 & 1.0 & 0.0 \end{pmatrix}.$$

One final option is $A = PLU$ with the same U, but with

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 2/5 & 1 & 0 \\ 3/5 & 2/3 & 1 \end{pmatrix}.$$

8. (20 points) Let

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 5 & -1 & -1 \\ 0 & 3 & -3 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 8 \\ -8 \\ 15 \end{pmatrix}.$$

Use the LU factorization $A = LU$ to solve the matrix equation $A\mathbf{x} = \mathbf{b}$.

If you compute A you will get 0 points!

answer: We want to solve $A\mathbf{x} = LU\mathbf{x} = \mathbf{b}$. So we define $\mathbf{y} = U\mathbf{x}$. Next we solve the equation $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$, and then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$.

The vector

$$\mathbf{y} = \begin{pmatrix} 8 \\ 0 \\ -1 \end{pmatrix}$$

is the solution of the system $L\mathbf{y} = \mathbf{b}$.

Then the vector

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

is the solution of $U\mathbf{x} = \mathbf{y}$ and so, of $A\mathbf{x} = \mathbf{b}$.

9. (15 points) Let $A = \begin{pmatrix} 5 & 4 \\ -9 & 1 \\ -7 & 7 \end{pmatrix}$.

Find A^T, the transpose of A.

answer: The transpose $A^T = \begin{pmatrix} 5 & -9 & -7 \\ 4 & 1 & 7 \end{pmatrix}$.

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10. (15 points) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

Find the inverse of A if it exists.

answer: $A^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$.

11. (15 points) Let $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Find the rank of the matrix A. Explain your answer.

answer: The rank of A is 0 because the rank is the dimension of the column space of A, and the column space is the zero subspace in this case.

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12. (15 points) Let $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$.

Are the given vectors linearly independent? Show your work and explain your answer.

answer: The vectors are linearly independent since (after constructing a matrix using the vectors as the columns) every column has a pivot.

13. (30 points) Let $A = \begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix}$.

(a) Find a nonzero vector in the column space of A. Show your work.

answer: I found the vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ in the column space. You'll perhaps find a different vector.

(b) Find a nonzero vector in the null space of A. Show your work.

answer: I found the vector $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ in the null space. You'll perhaps find a different vector.

14. (20 points) Let $A = \begin{pmatrix} 2 & -4 & -2 & 0 & 0 \\ 2 & -4 & -4 & -2 & 2 \\ 1 & -2 & 0 & 4 & -4 \\ 2 & -4 & -1 & -3 & 3 \end{pmatrix}$.

Find bases for $\mathrm{Col}(A)$ and $\mathrm{Nul}(A)$.

Hint: The reduced row echelon form of A is $\begin{pmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

answer: One basis for the column space of A is the set

$$\left\{ \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -4 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 4 \\ -3 \end{pmatrix} \right\}$$

One basis for the null space of A is the set

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$