Math 4350

Homework 5 (Gates for a single qubit)

(Due Wednesday, March 4)

Problem 1. Show that the following identities are correct.

$$XYX = -Y \qquad XR_y(\theta)X = R_y(-\theta)$$
$$XZX = -Z \qquad XR_z(\theta)X = R_z(-\theta)$$

Problem 2. Show that the following identities are correct.

$$HXH = Z \qquad HYH = -Y \qquad HZH = X$$

Problem 3. Express the Hadamard gate H as a product of R_x and R_z rotations and $e^{i\phi}$ for some real ϕ.

Problem 4. Find A, B, C, and α (from the Corollary) for the Hadamard gate.

Recall. We have the following gates:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/8} \end{bmatrix}$$

$$R_x(\theta) \equiv e^{-i(\theta/2)X} = \cos(\theta/2)I - i\sin(\theta/2)X = \begin{bmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{bmatrix}$$

$$R_y(\theta) \equiv e^{-i(\theta/2)Y} = \cos(\theta/2)I - i\sin(\theta/2)Y = \begin{bmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{bmatrix}$$

$$R_z(\theta) \equiv e^{-i(\theta/2)Z} = \cos(\theta/2)I - i\sin(\theta/2)Z = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

Some Notes

Lemma. *For any unitary gate U on a single quibit there are real numbers α, β, γ, and δ such that*

$$U = \begin{bmatrix} \exp(i(\alpha - \beta/2 - \delta/2)) \cos \gamma/2 & -\exp(i(\alpha - \beta/2 + \delta/2)) \sin \gamma/2 \\ \exp(i(\alpha + \beta/2 - \delta/2)) \sin \gamma/2 & \exp(i(\alpha + \beta/2 + \delta/2)) \cos \gamma/2 \end{bmatrix}$$

Proof. This follows from the fact that U has orthonormal rows and columns. □

Theorem. *For any unitary gate U on a single quibit there are real numbers α, β, γ, and δ such that*

$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$$

Proof. Just multiply and compare with the formula in the Lemma. □

Corollary. *For any unitary gate U on a single quibit there are unitary gates on a single qubit A, B, C such that $ABC = I$ and $U = e^{i\alpha} AXBXC$, where α is some real number.*

Proof. In the notation of the Theorem let

$$A \equiv R_z(\beta) R_y(\gamma/2), \quad B \equiv R_y(-\gamma/2) R_z(-(\delta+\beta)/2), \quad C \equiv R_z((\delta-\beta)/2).$$

Show $ABC = I$. Use Problem (1) to evaluate XBX. Use the fact that $R_y(\theta)R_y(\phi) = R_y(\theta+\phi)$ and $R_z(\theta)R_z(\phi) = R_z(\theta+\phi)$ to evaluate $AXBXC$. □

These problems and notes were take from *Quantum Computation and Quantum Information* by Nielsen and Chuang.

Solutions

Problem 3.

Express the Hadamard gate H as a product of R_x and R_z rotations and $e^{i\phi}$ for some real ϕ.

solution: Musing on the forms of R_z (diagonal) and R_x (made up of sines and cosines) it seems (somewhat) reasonable to start with $R_x(\pi/2)$ and try to adjust the entries by multiplying by some diagonal matrices on the left and right. If this works, then we can worry about writing the diagonal matrices we used in terms of R_z. Let's compare H and $R_x(\pi/2)$.

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$R_x(\pi/2) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

With luck a moment's reflection suggests

$$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} R_x(\pi/2) \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= H$$

Now write

$$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = e^{i\pi/4} \begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$
$$= e^{i\pi/4} R_z(\pi/2)$$

Assembling all the pieces we have

$$H = e^{i\pi/2} R_z(\pi/2) R_x(\pi/2) R_z(\pi/2)$$

Problem 4.

Find A, B, C, and α (from the Corollary) for the Hadamard gate.

solution: Using the statement of the Lemma, it's (more than) reasonable to set $\gamma = \pi/2$ so we have

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{i(\alpha-\beta/2-\delta/2)} & -e^{i(\alpha-\beta/2+\delta/2)} \\ e^{i(\alpha+\beta/2-\delta/2)} & e^{i(\alpha+\beta/2+\delta/2)} \end{bmatrix}$$

This leads to the system

$$\begin{aligned} \alpha - \beta/2 - \delta/2 &= 0 \qquad & \alpha - \beta/2 + \delta/2 &= \pi \\ \alpha + \beta/2 - \delta/2 &= 0 \qquad & \alpha + \beta/2 + \delta/2 &= \pi \end{aligned}$$

A little work shows that

$$\alpha = \pi/2 \quad \beta = 0 \quad \delta = \pi$$

The formula in the corollary then gives us

$$\begin{aligned} A &= R_z(\beta)R_y(\gamma/2) \\ &= R_z(0)R_y(\pi/4) \\ &= \begin{bmatrix} \cos(\pi/8) & -\sin(\pi/8) \\ \sin(\pi/8) & \cos(\pi/8) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} B &= R_y(-\gamma/2)R_z(-(\delta+\beta)/2) \\ &= R_y(-\pi/4)R_z(-\pi/2) \\ &= \begin{bmatrix} \cos(\pi/8) & \sin(\pi/8) \\ -\sin(\pi/8) & \cos(\pi/8) \end{bmatrix}\begin{bmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{bmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{bmatrix} (1+i)\cos(\pi/8) & (1-i)\sin(\pi/8) \\ -(1+i)\sin(\pi/8) & (1-i)\cos(\pi/8) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} C &= R_z((\delta-\beta)/2) \\ &= R_z(\pi/2) \\ &= \begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix} \end{aligned}$$

$$\alpha = \pi/2$$
