Math 4350

Homework 4 (Fourier Matrices)

What follows is an approach to showing that the Fourier matrices are feasible. We introduce some notation that builds on (and corrects) that in the book.

Definition 1. For $N = 2^n$, let $\omega = \omega_N = e^{2\pi i/N}$ and

$$G_N = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{N-2} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{N-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{N-2} & \omega^{N-3} & \cdots & \omega \end{bmatrix}$$

That is,

$$G_N[r, c] = \omega_N^{rc} = \omega_N^{rc \pmod N}.$$

Definition 2. For $N = 2^n$, the Fourier matrix F_N of order N is $\frac{1}{\sqrt{N}} G_N$.

Definition 3 (*altered from Problem 5.5*)**.** Let D_N be the diagonal matrix formed by the second column of G_N.

$$D_N = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \omega^3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \omega^{N-1} \end{bmatrix}$$

Definition 4. Let D' and D'' be the ${}^N\!/_2 \times {}^N\!/_2$ diagonal matrices so that

$$D_N = \begin{bmatrix} D' & 0 \\ 0 & D'' \end{bmatrix}$$

Definition 5 (*taken from Problem 4.9*)**.** Let K_n be the cycle matrix that (under the action of a matrix on the standard basis vectors e_x) takes the string $x_1 x_2 \cdots x_n$ to the string $x_2 \cdots x_n x_1$.

The notation given above is in force throughout the problems.

Problem 1. Show how to write D_N as a tensor product of the 2×2 twist matrices

$$T_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{bmatrix}$$

defined in problem 3.10. That is, find $\alpha_1, \ldots, \alpha_n$ so that $D_N = T_{\alpha_1} \otimes \cdots \otimes T_{\alpha_n}$.

Hint: Do the 2×2 case first, then the 4×4 case, etc. By the time you've done the 8×8 case you should see the pattern.

Problem 2. Show that $D'' = -D'$.

Hint: This is really just a question about ω_N^k for various k.

Problem 3. Consider the cycle matrix K_n

(a) For each row r find a formula/description for the (only) column c where K_n has a 1 in this row. Give your formula/description in terms of both the numbers r and c as well as in terms of their string representations.

(b) Repeat the exercise for the transpose K_n^T.

Hint: Start by looking at the 8×8 case. If the pattern isn't clear look at the 16×16 case, etc.

Problem 4 (*altered from 5.6*)**.** Show that G_N obeys the following recursive equation in block matrices:

$$G_N = \begin{bmatrix} I_{N/2} & D' \\ I_{N/2} & D'' \end{bmatrix} \begin{bmatrix} G_{N/2} & 0 \\ 0 & G_{N/2} \end{bmatrix} K_n^T$$

Hint: Multiply the 2×2 block matrices and then use Problem 3 to figure out how the columns of this block matrix are shuffled when multiplied on the right by by K_n^T. Compare with the columns of G_N.