Math 4350

Homework 2

(3.2) If U is a matrix, then what is $U\mathbf{e}_k$?

answer: The k^{th} column of U.

(3.4) Consider the matrix

$$U = \begin{bmatrix} w & w \\ w & -w \end{bmatrix}.$$

For what real values of w is it a unitary matrix?

solution: Since w is real the adjoint $U^* = U^T$ and clearly $U^T = U$. So this particular U is unitary iff $UU = I$. Let's take a look.

$$\begin{aligned} UU &= \begin{bmatrix} w & w \\ w & -w \end{bmatrix} \begin{bmatrix} w & w \\ w & -w \end{bmatrix} \\ &= \begin{bmatrix} 2w^2 & w^2 - w^2 \\ w^2 - w^2 & 2w^2 \end{bmatrix} \\ &= \begin{bmatrix} 2w^2 & 0 \\ 0 & 2w^2 \end{bmatrix} \end{aligned}$$

This is the identity iff $2w^2 = 1$. In othe words

$$w = \pm \frac{1}{\sqrt{2}}$$

(3.7) For any complex $N \times N$ matrix U, we can uniquely write $U = R + iQ$, where Q and R have real entries. Show that if U is unitary, then is the $2N \times 2N$ matrix V given in block form by

$$V = \begin{bmatrix} R & Q \\ -Q & R \end{bmatrix}.$$

Thus, by doubling the dimension, we can remove the need for complex-number entries.

solution: Since R and Q have real entries we know $\overline{R} = R$ and $\overline{Q} = Q$. Consequently the adjoint

$$\begin{aligned} U^* &= \overline{U}^T \\ &= \overline{R + iQ}^T \\ &= (R - iQ)^T \\ &= R^T - iQ^T \end{aligned}$$

Now U a $N \times N$ unitary matrix implies that

$$\begin{aligned} I_N &= U^*U \\ &= (R^T - iQ^T)(R + iQ) \\ &= R^TR + iR^TQ - iQ^TR - i^2Q^TQ \\ &= (R^TR + Q^TQ) + i(R^TQ - Q^TR) \end{aligned}$$

We see that

$$\begin{aligned} R^TR + Q^TQ &= I_N \\ R^TQ - Q^TR &= 0_N \end{aligned}$$

Now we compute V^*V making use of the fact that V is real so $V^* = V^T$.

$$\begin{aligned} V^*V &= V^TV \\ &= \begin{bmatrix} R & Q \\ -Q & R \end{bmatrix}^T \begin{bmatrix} R & Q \\ -Q & R \end{bmatrix} \\ &= \begin{bmatrix} R^T & -Q^T \\ Q^T & R^T \end{bmatrix} \begin{bmatrix} R & Q \\ -Q & R \end{bmatrix} \\ &= \begin{bmatrix} R^TR + Q^TQ & R^TQ - Q^TR \\ Q^TR - R^TQ & Q^TQ + R^TR \end{bmatrix} \\ &= \begin{bmatrix} I_N & 0_N \\ 0_N & I_N \end{bmatrix} \\ &= I_{2N} \end{aligned}$$

and V is unitary.

(3.8) Apply the construction of the last problem to the matrix

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

This is the second of the so-called *Pauli matrices.*

solution: We have

$$\begin{aligned} Y &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + i \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= R + iQ \end{aligned}$$

and so

$$\begin{aligned} Y' &= \begin{bmatrix} R & Q \\ -Q & R \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

(3.9) Consider the following matrix

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\pi/4} & e^{-i\pi/4} \\ e^{-i\pi/4} & e^{i\pi/4} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

What is V^2?

solution: We simply compute.

$$\begin{aligned} V^2 &= \frac{1}{2} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} (1+i)^2 + (1-i)^2 & (1+i)(1-i) + (1-i)(1+i) \\ (1-i)(1+i) + (1+i)(1-i) & (1-i)^2 + (1+i)^2 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 2i - 2i & 2+2 \\ 2+2 & -2i+2i \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

Or if you prefer to use the first representation of the matrix:

$$\begin{aligned} V^2 &= \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\pi/4} & e^{-i\pi/4} \\ e^{-i\pi/4} & e^{i\pi/4} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\pi/4} & e^{-i\pi/4} \\ e^{-i\pi/4} & e^{i\pi/4} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{i\pi/4}e^{i\pi/4} + e^{-i\pi/4}e^{-i\pi/4} & e^{i\pi/4}e^{-i\pi/4} + e^{-i\pi/4}e^{i\pi/4} \\ e^{-i\pi/4}e^{i\pi/4} + e^{i\pi/4}e^{-i\pi/4} & e^{-i\pi/4}e^{-i\pi/4} + e^{i\pi/4}e^{i\pi/4} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{i\pi/2} + e^{-i\pi/2} & e^0 + e^0 \\ e^0 + e^0 & e^{-i\pi/2} + e^{i\pi/2} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} i - i & 1+1 \\ 1+1 & -i+i \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$