Math 3200

Homework 2

(1.2) Use the prime enumeration sieve to make a list of all primes up to 100. *Use Algorithm 1.2.3 and show all of your work, displaying P and X at each step.*

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solution: I used a python program to do this problem. First the code.

```
# Algorithm 1.2.3
def prime_sieve(n):
    P = []
    X = range(2,n+1)
    while True:
        p=X[0]
        if p <= sqrt(n):
            P.append(p)
            X=[m for m in X if m%p != 0]
            print("P = {}\nX = {}\n".format(P,X))
        else:
            P.extend(X)
            print("P = {}\n".format(P))
            return P
```

Next the results from running `prime_sieve(100)`

```
P = [2]
X = [3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31,
    33, 35, 37, 39, 41, 43, 45, 47, 49, 51, 53, 55, 57, 59,
    61, 63, 65, 67, 69, 71, 73, 75, 77, 79, 81, 83, 85, 87,
    89, 91, 93, 95, 97, 99]

P = [2, 3]
X = [5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43,
    47, 49, 53, 55, 59, 61, 65, 67, 71, 73, 77, 79, 83, 85,
    89, 91, 95, 97]

P = [2, 3, 5]
X = [7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53,
    59, 61, 67, 71, 73, 77, 79, 83, 89, 91, 97]

P = [2, 3, 5, 7]
X = [11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61,
    67, 71, 73, 79, 83, 89, 97]
```

```
P = [2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47,
    53, 59, 61, 67, 71, 73, 79, 83, 89, 97]
```

(1.7) (a) Let $y = 10000$. Compute $\pi(y) = \#\{\text{primes} \le y\}$.

(b) The prime number theorem implies $\pi(x)$ is asymptotic to $\frac{x}{\log(x)}$. How close is $\pi(y)$ to $y/\pi(y)$, where y is as in (a)?

Also compare $\pi(y)$ *to* $\frac{y}{\log(y)-1}$ *for* $y = 10000$. *You may use a computer on this problem.*

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solution: I used my python program from the previous problem. For part (a) I got this. (I removed the printing lines before running it.)

```
>>> P = prime_sieve(10000)
>>> len(P)
1229
```

For part (b) I got this.

```
>>> import math
>>> 10000/math.log(10000)
1085.7362047581294
>>> 1229-10000/math.log(10000)
143.26379524187064
```

And then for the added question I got this.

```
>>> 10000/(math.log(10000)-1)
1217.97630146155
>>> 1229-10000/(math.log(10000)-1)
11.023698538449935
```

A much better approximation in this case.

(1.8) Let a, b, c, n be integers. Prove that

(a) if $a \mid n$ and $b \mid n$ with $\gcd(a, b) = 1$, then $ab \mid n$.

(b) if $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.

Feel free to use the Fundamental Theorem of Arithmetic on this problem.

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solution:

(a) Since $a \mid n$ we can write $n = ac$. Now by the Fundamental Theorem of Arithmetic (FTA) we can write b as a product of primes: $b = q_1 \cdots q_r$ and

$$q_1 \cdots q_r \mid ac$$

Since $\gcd(a, b) = 1$ none of the primes q_k divide a. By repeatedly using Theorem 1.1.19 to cancel the q_k we have (starting with $c_1 = c$)

$$\begin{array}{cccc} q_1 \mid c_1 & c_1 = q_1 c_2 & q_1 \cdots q_r \mid ac_1 = q_1 ac_2 & q_2 \cdots q_r \mid ac_2 \\ q_2 \mid c_2 & c_2 = q_2 c_3 & q_2 \cdots q_r \mid ac_2 = q_2 ac_3 & q_3 \cdots q_r \mid ac_3 \\ & \cdots & & \\ q_{r-1} \mid c_{r-1} & c_{r-1} = q_{r-1} c_r & q_{r-1} q_r \mid ac_{r-1} = q_{r-1} ac_r & q_r \mid ac_r \\ q_r \mid c_r & c_r = q_r c_{r+1} & & \end{array}$$

So we have

$$\begin{aligned} c &= c_1 \\ &= q_1 c_2 \\ &= q_1 q_2 c_3 \\ &\cdots \\ &= q_1 q_2 \cdots q_r c_{r+1} \\ &= bc_{r+1} \end{aligned}$$

and

$$n = ac = abc_{r+1} \qquad \text{i.e., } ab \mid n$$

(b) Let $n = bc$. Then $a \mid n$ and $b \mid n$ with $\gcd(a, b) = 1$. Applying part (a) gives us $ab \mid n = bc$, so cancelling b gives us $a \mid c$.