Chapter 6

Math 2890-001
Fall 2017 **Name** ____________________
Due Nov 06

1. (1 point) Let

$$u = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 0 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 7 \\ 3 \\ 4 \\ 1 \end{pmatrix}.$$

Find the inner product $u \cdot v$. Show your work.

answer: $u \cdot v = \sum_k u_k v_k = (1)(7) + (3)(3) + (5)(4) + (0)(1) = 36.$

2. (1 point) Let

$$v = \begin{pmatrix} 6 \\ -4 \\ -2 \\ 5 \end{pmatrix}.$$

Find a unit vector in the direction of v. Show your work.

answer: Since $\|v\|^2 = v \cdot v = (6)^2 + (-4)^2 + (-2)^2 + (5)^2 = 81$, the unit vector

$$u = \pm\frac{v}{\|v\|} = \pm \begin{pmatrix} 6 \\ -4 \\ -2 \\ 5 \end{pmatrix} \frac{1}{9} = \pm \begin{pmatrix} 0.6667 \\ -0.4444 \\ -0.2222 \\ 0.5556 \end{pmatrix}.$$

3. (1 point) Let

$$u = \begin{pmatrix} 5 \\ 4 \\ 3 \\ 6 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 3 \\ -6 \\ -4 \\ 2 \end{pmatrix}.$$

Find the distance between u and v. Show and explain your computations.

answer: The distance is $\|u - v\| = \sqrt{169} = 13$, where $u - v = \begin{pmatrix} 2 \\ 10 \\ 7 \\ 4 \end{pmatrix}$.

4. (1 point) Let

$$u_1 = \begin{pmatrix} 1 \\ -5 \\ 5 \\ -1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 5 \\ -3 \\ -3 \\ 5 \end{pmatrix} \quad \text{and} \quad u_3 = \begin{pmatrix} 7 \\ 7 \\ 5 \\ -3 \end{pmatrix}.$$

Is the set $\{u_1, u_2, u_3\}$ orthogonal? Why or why not? Show your computations.

answer: No, $u_2 \cdot u_3 = -16 \neq 0$.

5. (1 point) Let

$$y = \begin{pmatrix} 2 \\ 0 \\ 7 \\ 1 \end{pmatrix}$$

and let W be the span of

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}.$$

Project y onto W. Show and explain your computations.

answer: The projection is

$$\begin{aligned} w &= A(A^T A)^{-1}(A^T y) \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 6 & 15 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 12 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 1 \\ 2 \\ -3 \end{pmatrix} \end{aligned}$$

6. (1 point) Let

$$y = \begin{pmatrix} 2 \\ 0 \\ 7 \\ 1 \end{pmatrix}$$

and let W be the span of

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}.$$

Find the point in W that is closest to y. Show and explain your computations.

answer: The closest point is the projection of y to W:

$$\begin{aligned} w &= A(A^T A)^{-1}(A^T y) \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 6 & 15 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 12 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 1 \\ 2 \\ -3 \end{pmatrix} \end{aligned}$$

7. (1 point) Let

$$y = \begin{pmatrix} 2 \\ 0 \\ 7 \\ 1 \end{pmatrix}$$

and let W be the span of

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}.$$

Write y as a sum of a vector in W and a vector orthogonal to W. Show and explain your computations.

answer: The projection is

$$\begin{aligned} w &= A(A^T A)^{-1}(A^T y) \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 6 & 15 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 12 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 1 \\ 2 \\ -3 \end{pmatrix} \end{aligned}$$

The vector orthogonal to W is

$$v = y - w = \begin{pmatrix} 2 \\ 0 \\ 7 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 5 \\ 4 \end{pmatrix}.$$

We have $y = \begin{pmatrix} 2 \\ 0 \\ 7 \\ 1 \end{pmatrix} = w + v = \begin{pmatrix} -1 \\ 1 \\ 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 5 \\ 4 \end{pmatrix}.$

8. (1 point) Let

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 2 \\ 1 & 2 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 7 \\ 7 \end{pmatrix}.$$

Find the least squares solution to $Ax = b$. Show and explain your computations.

answer: The normal equations $A^T Ax = A^T b$

are $\begin{pmatrix} 8 & 12 \\ 12 & 19 \end{pmatrix} x = \begin{pmatrix} 4 \\ 13 \end{pmatrix}$,

and these have solution $x = \begin{pmatrix} -10 \\ 7 \end{pmatrix}$.

9. (1 point) Let

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 2 \\ 1 & 2 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 7 \\ -2 \\ 1 \\ 2 \\ 2 \end{pmatrix}.$$

Find the least squares error in the least squares solution to $Ax = b$. Show and explain your computations.

HINT: The least squares solution is $x = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

answer: The least squares error is $\|Ax - b\| = \sqrt{59}$,

$$\text{since } Ax - b = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 7 \\ -2 \\ 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \\ -1 \\ -2 \\ -3 \end{pmatrix}.$$

10. (1 point) Let

$$Q = \begin{pmatrix} 2/9 & 4/9 \\ -5/9 & 6/9 \\ -4/9 & 2/9 \\ 6/9 & 5/9 \end{pmatrix}, \quad R = \begin{pmatrix} 5 & -11 \\ 0 & 3 \end{pmatrix} \text{ and } b = \begin{pmatrix} 11 \\ -1 \\ 3 \\ 2 \end{pmatrix}.$$

Use the QR factorization $A = QR$ to find the least squares solution to $Ax = b$.

Show your work.

answer: The equation $Rx = Q^T b$ is $\begin{pmatrix} 5 & -11 \\ 0 & 3 \end{pmatrix} x = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$,

and this has solution $x = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

11. (1 point) Let

$$A = \begin{pmatrix} 0 & -1 & 6 \\ -1 & 4 & -9 \\ -3 & 9 & -12 \\ 1 & -3 & 3 \\ 1 & -2 & 0 \end{pmatrix}.$$

Find the QR factorization of A.

Show and explain your computations.

answer:

$$Q = \begin{pmatrix} 0 & -1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{12} & 1/\sqrt{3} & 0 \\ -3/\sqrt{12} & 0 & 0 \\ 1/\sqrt{12} & 0 & -1/\sqrt{3} \\ 1/\sqrt{12} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} 0 & -0.5774 & 0.5774 \\ -0.2887 & 0.5774 & 0 \\ -0.866 & 0 & 0 \\ 0.2887 & 0 & -0.5774 \\ 0.2887 & 0.5774 & 0.5774 \end{pmatrix}$$

and

$$R = \begin{pmatrix} 12/\sqrt{12} & -36/\sqrt{12} & 48/\sqrt{12} \\ 0 & 3/\sqrt{3} & -15/\sqrt{3} \\ 0 & 0 & 3/\sqrt{3} \end{pmatrix} = \begin{pmatrix} 3.4641 & -10.3923 & 13.8564 \\ 0 & 1.7321 & -8.6603 \\ 0 & 0 & 1.7321 \end{pmatrix}$$

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12. (1 point) Use the QDR factorization

$$A = \begin{pmatrix} -2 & 0 & 8 \\ -1 & -2 & -6 \\ -2 & -4 & -7 \\ 1 & 0 & -4 \end{pmatrix} = \underbrace{\begin{pmatrix} -2 & 2 & 0 \\ -1 & -1 & -2 \\ -2 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} 1/10 & 0 & 0 \\ 0 & 1/10 & 0 \\ 0 & 0 & 1/5 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 10 & 10 & 0 \\ 0 & 10 & 40 \\ 0 & 0 & 5 \end{pmatrix}}_{R}$$

to find the least squares solution to $Ax = b$ where $b = \begin{pmatrix} 10 \\ -5 \\ 10 \\ -15 \end{pmatrix}$.

answer: Since $Q^T A = Q^T QDR = R$, we just need to solve the equation $Rx = Q^T b$.

This equation is $\begin{pmatrix} 10 & 10 & 0 \\ 0 & 10 & 40 \\ 0 & 0 & 5 \end{pmatrix} x = \begin{pmatrix} -50 \\ 20 \\ 20 \end{pmatrix}$, and has solution $x = \begin{pmatrix} 9 \\ -14 \\ 4 \end{pmatrix}$.

13. (1 point) Let

$$A = \begin{pmatrix} -2 & 4 & -12 \\ 3 & -2 & -1 \\ 0 & 2 & -7 \\ 1 & 0 & -1 \\ -4 & 4 & -10 \end{pmatrix}.$$

Find the QDR factorization of A.

Show and explain your computations.

answer: Use Wedderburn rank reduction.

$$Q = \begin{pmatrix} -2 & 2 & 0 \\ 3 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \\ -4 & 0 & -2 \end{pmatrix}$$

$$D = \begin{pmatrix} 1/30 & 0 & 0 \\ 0 & 1/10 & 0 \\ 0 & 0 & 1/15 \end{pmatrix}$$

$$R = \begin{pmatrix} 30 & -30 & 60 \\ 0 & 10 & -40 \\ 0 & 0 & 15 \end{pmatrix}$$

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14. (1 point) Consider the data points $(1, -3), (2, -6), (3, 9), (4, 1)$.

Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that best fits the given data points.

Show and explain your computations.

answer: I need to find the least squares solution of

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \\ 9 \\ 1 \end{pmatrix}.$$

The normal equations are $\begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 16 \end{pmatrix}$, and these have solution $\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} -6.5 \\ 2.7 \end{pmatrix}$. The least squares line is

$$y = -6.5 + 2.7x$$

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15. (1 point) Consider the data points $(1, -5), (2, -7), (3, -4), (4, 5)$.

Find the equation $y = \beta_0 + \beta_1 x + \beta_2 x^2$ of the least-squares quadratic that best fits the given data points.

Show and explain your computations.

answer: I need to find the least squares solution of

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} -5 \\ -7 \\ -4 \\ 5 \end{pmatrix}.$$

The normal equations are $\begin{pmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} -11 \\ -11 \\ 11 \end{pmatrix}$, and these have solution $\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 2.75 \\ -10.45 \\ 2.75 \end{pmatrix}$. The least squares curve is

$$y = 2.75 - 10.45x + 2.75x^2$$

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