Number of different binary trees

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- This assignment is due Thursday, October 21.

Let b_n denote the number of different binary trees with n nodes. In these problems you will find a formula for b_n, as well as an asymptotic estimate.

1. Show that $b_0 = 1$ and that, for $n \geq 1$

$$b_n = \sum_{k=0}^{n-1} b_k \, b_{n-1-k}.$$

2. Let $B(x)$ be the *generating function* (formal power series)

$$B(x) = \sum_{n=0}^{\infty} b_n \, x^n.$$

   Show that $B(x) = xB(x)^2 + 1$, and hence one way to express $B(x)$ in closed form is

$$B(x) = \frac{1}{2x}\left(1 - \sqrt{1-4x}\,\right).$$

3. Show that

$$b_n = \frac{1}{n+1}\binom{2n}{n}.$$

   (the n^{th} *Catalan number*) by using either

   (a) the Taylor expansion of $\sqrt{1-4x}$ around $x = 0$, or

   (b) the generalization of the binomial expansion to nonintegral exponents.

   In either case, first clearly state the formula for the expansion you are using and then apply it to the function $B(x)$.

4. Find and use *Stirling's approximation* for $n!$ to show that

$$b_n = \frac{4^n}{\sqrt{\pi}\, n^{3/2}}\left(1 + O\left({}^{1}\!/\!{}_{n}\right)\right).$$