Matrix Factorizations

We are now done with the course and we have six factorizations. Three are used to solve linear systems while three are used to solve eigensystems.

Linear Systems: $Ax = b$

Given matrix $A_{m\times n}$ and vector $b_{m\times 1}$ find vectors $x_{n\times 1}$ that satisfy equation.

- A is invertible (most square matrices, i.e., $m = n$).

  $A = LDU$

  $L_{m\times m}, U_{m\times m}$ triangular, $D_{m\times m}$ diagonal

- A has linearly independent columns (most matrices with $m \geq n$).

  $A = QDR$

  $R_{n\times n}$ triangular, $D_{n\times n}$ diagonal, $Q_{m\times n}$ with $Q^T QD = I_{n\times n}$

- A is any matrix.

  $A = U\Sigma V^*$ (Full Singular Value Decomposition)

  $U_{m\times m}, V_{n\times n}$ unitary, $\Sigma_{m\times n}$ diagonal with non-negative entries

  $A = \hat{U}\hat{\Sigma}\hat{V}^*$ (Reduced Singular Value Decomposition)

  $\hat{U}_{m\times r}, \hat{V}_{n\times r}$ with $(\hat{U}^*)(\hat{U}) = (\hat{V}^*)(\hat{V}) = I_{r\times r}$

  $\hat{\Sigma}_{r\times r}$ invertible, diagonal with non-negative entries

Eigen Systems: $Ax = x\lambda$

Given square matrix $A_{m\times m}$ find scalars λ and vectors $x_{m\times 1} \neq 0$ that satisfy equation.

- A is normal ($AA^* = A^*A$).

  $A = UDU^*$

  $U_{m\times m}$ unitary, $D_{m\times m}$ diagonal

- A is diagonalizable (most square matrices).

  $A = PDP^{-1}$

  $P_{m\times m}$ invertible, $D_{m\times m}$ diagonal

- A is any square matrix.

  $A = UTU^*$

  $U_{m\times m}$ unitary, $T_{m\times m}$ triangular