Problem:

Evaluate

$$\iint_A x \; dx \, dy$$

where A is the region in the first quadrant bounded by the curves

$$\begin{aligned} y + x^2 &= 4, \quad y - x^2 = 0 \\ y + x^2 &= 8, \quad y - x^2 = 2 \end{aligned}$$

Solution:

You should first sketch the region so you're clear on what's going on. We'll use the transformation

$$u = y + x^2 \qquad v = y - x^2$$

Solving for x and y we find the inverse transformation

$$x = \sqrt{\frac{u-v}{2}} \qquad y = \frac{u+v}{2}$$

Our integral becomes

$$\int_0^2 \int_4^8 \sqrt{\frac{u-v}{2}} \, \frac{\partial(x,y)}{\partial(u,v)} \, du \, dv \tag{1}$$

Now to compute the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{4}\sqrt{\frac{2}{u-v}} & \frac{-1}{4}\sqrt{\frac{2}{u-v}} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4}\sqrt{\frac{2}{u-v}}$$

Substituting this into (1) gives us the integral

$$\int_0^2 \int_4^8 \frac{1}{4} \, du \, dv = 2$$