Exam 2
Math 2850-002
Fall 2013 **Name** ______________________________
Odenthal

Instructions: No books. No notes. Non-graphing calculators only. Please write neatly. There are 7 problems on 6 pages worth a total of 200 points. **Show your work! Explain your answers.**

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1. Suppose that f is a function of two variables such that $f(2,3) = 17$ and the gradient $\nabla f(2,3) = \langle 4, -5 \rangle$.

   (a) (15 points) Find the Linearization of f at the point $(x, y) = (2, 3)$

   answer:

$$\begin{aligned} L(x,y) &= f(2,3) + \nabla f(2,3) \cdot \langle x-2, y-3 \rangle \\ &= 17 + (4)(x-2) + (-5)(y-3) \end{aligned}$$

   (b) (15 points) Use the answer from part (a) to find an approximation for $f(2.3, 2.8)$.

   answer:

$$\begin{aligned} f(2.3, 2.8) &\approx L(2.3, 2.8) \\ &= 17 + (4)(2.3-2) + (-5)(2.8-3) \\ &= 19.2 \end{aligned}$$

2. Suppose that f is a function of three variables such that $f(1,3,2) = 15$ and the gradient $\nabla f(1,3,2) = \langle 7,\ 8,\ 9\rangle$.

   (a) (15 points) Compute the directional derivative of f at the point $P(1,3,2)$ in the direction of the vector $\mathbf{b} = \langle -2,\ 2,\ 1\rangle$.

   answer: First compute $|\mathbf{b}| = \sqrt{(-2)^2 + (2)^2 + (1)^2} = 3$.
   Letting $\mathbf{u} = \dfrac{\mathbf{b}}{|\mathbf{b}|} = \langle \dfrac{-2}{3}, \dfrac{2}{3}, \dfrac{1}{3}\rangle$ the desired directional derivative is

$$\begin{aligned} D_{\mathbf{u}} f(P) &= \nabla f(P) \cdot \mathbf{u} \\ &= \langle 7, 8, 9\rangle \cdot \langle \frac{-2}{3}, \frac{2}{3}, \frac{1}{3}\rangle \\ &= (7)\left(\frac{-2}{3}\right) + (8)\left(\frac{2}{3}\right) + (9)\left(\frac{1}{3}\right) \\ &= \frac{11}{3} \end{aligned}$$

   (b) (15 points) Find the equation of the tangent plane to the level surface $f(x,y,z) = 15$ at the point $P(1,3,2)$.

   answer: The points $Q(x,y,z)$ on the tangent plane satisfy

$$\nabla f(P) \cdot (Q - P) = 0$$

   This, of course, is just

$$7(x-1) + 8(y-3) + 9(z-2) = 0$$

3. Consider the function $f(x, y) = x^3 + 6xy - 3y^2$. Observe that f has the gradient $\nabla f(x, y) = \langle 3x^2 + 6y,\, 6x - 6y\rangle$.

(a) (15 points) Find all the critical points of f.

answer: We want the points where the gradient ∇f is zero. Since

$$\begin{aligned} f_x &= 3x^2 + 6y \\ f_y &= 6x - 6y \end{aligned}$$

we want to solve the system

$$\begin{aligned} 3x^2 + 6y &= 0 \\ 6x - 6y &= 0 \end{aligned} \qquad \text{which reduces to} \qquad \begin{aligned} x &= y \\ x^2 + 2x &= 0 \end{aligned}$$

This system has the two solutions $(x, y) = (0, 0)$ and $(x, y) = (-2, -2)$. These are the sought after critical points.

(b) (15 points) For each critical point you found in part (a) determine whether f has a local minimum, a local maximum, or a saddle point there.

answer: We'll apply the second derivative test.

$$\begin{aligned} f_{xx} &= 6x \\ f_{xy} &= 6 \\ f_{yy} &= -6 \end{aligned}$$

So the discriminant $f_{xx}f_{yy} - f_{xy}^2 = -36(x + 1)$.

(x, y)	$f_{xx}f_{yy} - f_{xy}^2$	classification
$(0, 0)$	-36	saddle point
$(-2, -2)$	36	local maximum since $f_{xx} = -12 < 0$

4. (20 points) Use Lagrange multipliers to find the extreme values (and where they occur) of the function $f(x, y) = 10x + 4y$ subject to the constraint $x^2 + y^2 = 29$.

answer: Defining $g(x, y) = x^2 + y^2$ and setting $\nabla f = \lambda \nabla g$ we get the system:

$$\begin{aligned} 10 &= \lambda 2x \\ 4 &= \lambda 2y \\ g(x, y) &= 29 \end{aligned}$$

Solving the first two equations for x and y in terms of λ yields $x = 5/\lambda$ and $y = 2/\lambda$. Putting these into the constraint gives us $\lambda^2 = 1$. This has the two solutions $\lambda = \pm 1$.

x	y	$f(x, y)$
5	2	58
-5	-2	-58

So f has a global *maximum* of 58 at $(x, y) = (5, 2)$, and a global *minimum* of -58 at $(x, y) = (-5, -2)$.

5. Compute the following partial derivatives.

(a) (15 points) $\dfrac{\partial f}{\partial x}$ for $f(x,y) = \ln(x^3y+1)$

answer:

$$\frac{\partial f}{\partial x} = \frac{3x^2y}{x^3y+1}$$

(b) (15 points) $\dfrac{\partial f}{\partial y}$ for $f(x,y) = e^{x^3y+1}$

answer:

$$\frac{\partial f}{\partial y} = x^3e^{x^3y+1}$$

(c) (15 points) $\dfrac{\partial f}{\partial y}$ for $f(x,y) = x^2 + xy^2 + y^3$

answer:

$$\frac{\partial f}{\partial y} = 2xy + 3y^2$$

(d) (15 points) $\dfrac{\partial f}{\partial y}$ for $f(x,y) = \sin\left(\dfrac{y}{x}\right)$

answer:

$$\frac{\partial f}{\partial y} = \frac{1}{x}\cos(y/x)$$

6. Suppose that $f(x,y)$ is a function of two variables with the following values for itself and its partial derivatives at the point $(0,0)$.

$$\begin{array}{llll} f(0,0)=3 & f_x(0,0)=2 & f_y(0,0)=-3 & \\ f_{xx}(0,0)=5 & f_{xy}(0,0)=4 & f_{yy}(0,0)=6 & \\ f_{xxx}(0,0)=1 & f_{xxy}(0,0)=7 & f_{xyy}(0,0)=-6 & f_{yyy}(0,0)=-2 \end{array}$$

Suppose further that the following bounds hold on the rectangle R defined by $|x| \le 0.2$ and $|y| \le 0.3$.

$$\begin{array}{llll} |f| \le 3.5 & |f_x| \le 2.4 & |f_y| \le 3.3 & \\ |f_{xx}| \le 5.2 & |f_{xy}| \le 4.5 & |f_{yy}| \le 8.6 & \\ |f_{xxx}| \le 1.7 & |f_{xxy}| \le 7.2 & |f_{xyy}| \le 6.8 & |f_{yyy}| \le 2.3 \end{array}$$

(a) (15 points) Write out the Taylor polynomial for f having degree 2 and centered at $(0,0)$.

answer:

$$\begin{aligned} p(x,y) &= f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}\left(f_{xx}(0,0)x^2 + f_{xy}(0,0)2xy + f_{yy}(0,0)y^2\right) \\ &= 3 + (2)x + (-3)y + \frac{1}{2}\left((5)x^2 + (4)2xy + (6)y^2\right) \\ &= 3 + 2x - 3y + \frac{5}{2}x^2 + 4xy + 3y^2 \end{aligned}$$

(b) (15 points) Find a bound for the error in using the polynomial from part (a) in place of f on the rectangle R.

answer: Since $|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}| \le M = 7.2$ on R and for any point (a,b) in the rectangle R we have $|a| \le 0.2$ and $|b| \le 0.3$. Then for some point (a,b) on R we have

$$\begin{aligned} \text{Error} &= \frac{1}{3!}\left(f_{xxx}(a,b)a^3 + f_{xxy}(a,b)3a^2b + f_{xyy}(a,b)3ab^2 + f_{yyy}(a,b)b^3\right) \\ &\le \frac{1}{3!}\left(|a|^3 + 3|a|^2|b| + 3|a||b|^2 + |b|^3\right) M \\ &\le \frac{1}{3!}\left(|a| + |b|\right)^3 M \\ &\le \frac{1}{3!}\left(0.2 + 0.3\right)^3 (7.2) \\ &= 0.15 \end{aligned}$$