2.1 Matrix Operations

Remark. Recall that there is a one-to-one correspondence between $m \times n$ matrices with entries in $\mathbb{R}$ and linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

So we can identify a linear transformation with its corresponding matrix and we'll denote a linear transformation by L_A where $L_A(x) = Ax$ for the matrix A. Also recall that we can determine the columns of A by

$$A_{*k} = L_A(I_{*k})$$

where I is the $n \times n$ identity matrix.

Matrix Addition

Remark. From last time we know that if A and B are matrices then

(1) the function sum $L_A + L_B$ is defined iff A and B are the same size, and
(2) in this case $L_A + L_B$ is also a linear transformation.

Definition. If A and B are two matrices having the same size then the *sum* $A+B$ is defined to be the matrix that satisfies the identity

$$L_{A+B} = L_A + L_B$$

Calculation.

$$(A+B)_{*k} = (L_A + L_B)(I_{*k}) = L_A(I_{*k}) + L_B(I_{*k}) = A_{*k} + B_{*k}$$
$$(A+B)_{ij} = A_{ij} + B_{ij}$$

Warning. If two matrices aren't the same size then their sum is **not** defined.

Matrix Multiplication

Remark. From last time we know that if A and B are matrices then

(1) the function compositon $L_A \circ L_B$ is defined iff the number of *columns* of A equals the number of *rows* of B, and
(2) in this case $L_A \circ L_B$ is also a linear transformation.

Definition. If A and B are two matrices with the number of columns of A equaling the number of rows of B then the *product* AB is defined to be the matrix that satisfies the identity

$$L_{AB} = L_A \circ L_B$$

Calculation.

$$(AB)_{*k} = (L_A \circ L_B)(I_{*k}) = L_A(L_B(I_{*k})) = A(B_{*k})$$

Observation. $A_{m\times n}B_{p\times q}$ is defined iff $n = p$ and in this case AB is $m \times q$.

Properties of Matrix Operations

Remark. Because we defined addition and multiplication of matrices in terms of addition and composition of functions we get all the properties of these operations on functions for free, so if the following products and sums are defined we have:

(1) $(A+B)+C = A+(B+C)$
(2) $(AB)C = A(BC)$
(3) $(A+B)C = AC+BC$
(4) $A(B+C) = AB+AC$
(5) $0_{m\times n}$ (from the zero function $\mathbb{R}^n \to \mathbb{R}^m$)
(6) $I_{n\times n}$ (from the identity function $\mathbb{R}^n \to \mathbb{R}^n$)
(7) $A+B = B+A$ (only for addition)

Warning. Matrix multiplication has a few quirks:

- noncommutative
- zero divisors
- no cancellation

The matrices $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are enough to illustrate these quirks.

Numerical Notes.

- Standard LAPACK computes AB by columns. There is a version written in C++ that calculates AB by rows.
- The computation of AB lends itself well to parallel processing. Give each processor access to A and to a column (or set of columns) of B. That's all the processor needs to compute the corresponding column(s) of AB.

Powers of a Matrix

$A^k = A \cdot A \cdots A$

Transpose of a Matrix

The rows of A^T are the columns of A "laid down". The columns of A^T are the rows of A "stood up".

Theorem 1 (Properties of the Transpose)**.** *Compare with the properties of the inverse in Section 2.2.*

- $(A^T)^T = A$
- $(AB)^T = B^T A^T$
- $(A+B)^T = A^T + B^T$