Section 1.3: Vector Equations

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Outline

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   - What we have done up until now.
   - What we need to do now.
   - Digression

2. Vectors
   - Parallel Arithmetic
   - Application to Linear Systems
   - Geometric interpretation

What we have done up until now.

- Given a linear system we can efficiently and accurately determine if it is consistent, i.e. has solution(s), by putting the corresponding augmented matrix into row echelon form using Gaussian Elimination with Partial Pivoting.
- If the system is, in fact, consistent we can then find the solution(s) by putting this row echelon form matrix into reduced row echelon form using Back Substitution.
- So why aren't we done with linear systems?

What's left to do?

- Our system may be inconsistent only due to errors in the entries of our equations. (These may be because of errors in our measurements or simply because of the deficiencies of the floating point number system.)
- Since the correct system does have a solution, our task is to find an approximation to it using our incorrect equations. We need to find a new way to look at linear systems.

Example

During the first 41 datys of 1801, the Italian astronomer Piazzi discovered and followed Ceres (the first discovered asteroid). After this it disappeared behind the sun after only being tracked for 9 degrees of its orbit. The observations led to an inconsistent linear system describing the orbit of Ceres.

Gauss (using ideas that we will investigate) calculated the orbit with amazing accuracy. After reemerging from behind the sun, Ceres was located in almost exactly the position Gauss predicted.

Floating Point Numbers

from "Text Processing in Python" by David Mertz

Floating point math is harder than you think! If you think you understand just how complex IEEE 754 math is, you are not yet aware of all of the subtleties.

By way of indication, Python luminary and erstwhile professor of numeric computing Alex Martelli commented in 2001 (on ⟨`comp.lang.python`⟩):

- Anybody who thinks he knows what he's doing when floating point is involved is either naive, or Tim Peters.

To which fellow Python guru Tim Peters responded:

- I find it's possible to be both (wink).

The trick about floating point numbers is that operations on them do not obey the arithmetic rules we learned in middle school: associativity, transitivity, commutativity. For example:

$$\ggg 7 == 7/25 * 25$$

False

$$\ggg 7 == 7/24 * 24$$

True

Definitions

We're going to start with a naive approach to vectors.

Definition

A (column) vector is a stack of numbers.

Examples

$$\begin{pmatrix} 8 \\ 2 \\ 3 \\ 5 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 9 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 7 \\ 3 \\ 5 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

The above vectors are said to be in $\mathbb{R}^5$, $\mathbb{R}^3$, $\mathbb{R}^4$, and $\mathbb{R}^2$ respectively.

Caveat

There are more abstract (and more general) ways to define vectors. We may touch on these at some point, but the above definition will usually suffice for us.

Definitions, continued.

Addition

Two vectors of the same height can be added componentwise:

$$\begin{pmatrix}3\\5\\2\end{pmatrix} + \begin{pmatrix}2\\1\\6\end{pmatrix} = \begin{pmatrix}3+2\\5+1\\2+6\end{pmatrix} = \begin{pmatrix}5\\6\\8\end{pmatrix}$$

Two vectors of different heights cannot be added.

Multiplication by a scalar

A vector can be multiplied by a scalar componentwise.

$$\begin{pmatrix}3\\5\\2\end{pmatrix} \cdot 6 = \begin{pmatrix}3\cdot 6\\5\cdot 6\\2\cdot 6\end{pmatrix} = \begin{pmatrix}18\\30\\12\end{pmatrix}$$

Why not multiply two vectors?

Those of you who have seen cross-products in Calc III may be surprised that we aren't going to define a vector product of two vectors. It turns out that defining such a vector product is only possible in $\mathbb{R}^3$ and, even there, is of no real use to us in this course.

Properties

Since all we are really doing with these vector operations is ordinary arithmetic in parallel, all of the usual arithmetic properties are satisfied. Nothing interesting.

Associativity of Addition

For $x, y, z \in \mathbb{R}^n$: $\quad (x + y) + z = x + (y + z)$

Commutativity of Addition

For $x, y \in \mathbb{R}^n$: $\quad x + y = y + x$

Associativity and Commutativity of Multiplication by Scalars

For $x \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$: $\quad (x\alpha)\beta = x(\alpha\beta) = x(\beta\alpha) = (x\beta)\alpha$

Distributivity of Multiplication over Addition

For $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$: $\quad (x + y)\alpha = x\alpha + y\alpha$
For $x \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$: $\quad x(\alpha + \beta) = x\alpha + x\beta$

Translating a Linear System as a Vector Equation

Consider the following equivalent equations:

$$\begin{aligned} 2x_1 - 5x_2 &= 9 \\ 3x_1 + 4x_2 &= 7 \end{aligned}$$

$$\begin{pmatrix} 2x_1 - 5x_2 \\ 3x_1 + 4x_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} 2x_1 \\ 3x_1 \end{pmatrix} + \begin{pmatrix} -5x_2 \\ 4x_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} x_1 + \begin{pmatrix} -5 \\ 4 \end{pmatrix} x_2 = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$$

This last is a so-called vector equation and these will be crucial throughout the rest of the course.

Geometry in $\mathbb{R}^n$

We'll describe a way to give a geometric interpretation of vectors and vector equations in $\mathbb{R}^2$ and $\mathbb{R}^3$; and even in $\mathbb{R}^n$ for $n > 3$ with a little(?) imagination.

The idea is that the geometric interpretation will give us inspiration for new algorithms to deal with inconsistent systems. No kidding.

But, as my computer drawing skills are extremely pitiful, I think it'll be best if – I may regret this – all the drawing is done by hand ...