STATEMENT OF RESEARCH

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1. INTRODUCTION

My primary field of research is Complex Analysis, with a specialization in Operator Theory. This includes studying properties of Hankel, Toeplitz, and composition operators on spaces of analytic functions. My thesis focused on the invertibility of Toeplitz operators with either nonnegative, bounded symbols or Carleson measure symbols on the weighted Bergman space of the unit disc via the Berezin transforms of their symbols. Additionally, I studied the same question on Bargmann-Fock space, as well as on certain Model subspaces of the Hardy space of the unit circle. Characterizing the invertibility of Toeplitz operators in terms of properties of the Berezin transforms of their symbols would be profound, since we would then know that the symbol is the gatekeeper to invertibility, rather than the operator itself. This would have applications to physics, where the Berezin transform is an important object of study.

2. UNDERGRADUATE RESEARCH PLANS

While most of my work involves post-graduate material, I seek to use my research experience and mathematical knowledge to gently introduce undergraduates to the concepts of Operator Theory, Complex Analysis, and Functional Analysis. For each interested student, I will implement detailed, personalized study plans and weekly one-on-one meetings culminating in short departmental presentations over a wide variety of topics within these fields. In this way, undergraduates interested in mathematical research will learn how to read papers, study advanced texts, type in LaTeX, format and write research, and present findings, all while learning a substantial amount of new material at a comfortable pace and spreading their newfound knowledge to their colleagues and friends. In this way, I will create a self-sustaining, self-improving system of undergraduate research that, by construction, attracts youthful, curious mathematical talent.

3. BACKGROUND AND PRIOR RESEARCH

Define the unit disc $\mathbb{D}$ to be $\{z \in \mathbb{C} : |z| < 1\}$. For $\alpha > -1$, we denote by $A^2_\alpha(\mathbb{D})$ the weighted Bergman space of the unit disc, which is the set of all holomorphic functions on the disc which are square integrable with respect to the weighted Lebesgue area measure $dA_{\alpha}(z) := (1 - |z|^2)^{2+\alpha}dA(z)$. It is well known that the Bergman space is a closed subspace of $L^2_\alpha(\mathbb{D})$. As such, $A^2_\alpha(\mathbb{D})$ is a reproducing kernel Hilbert space with kernel function $K_\alpha(z,a)$ for each fixed $a \in \mathbb{D}$ called the Bergman kernel. For $z, a \in \mathbb{D}$, $K^\alpha_{\alpha}(z) :=$

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The normalized Bergman kernel is given by 
\[ K_a(z, a) = \frac{1}{(1 - \pi z)^{2+\alpha}}. \]
Additionally, there must exist an orthogonal projection 
\[ P_a : L^2_a(\mathbb{D}) \rightarrow A^2_a(\mathbb{D}) \]
known as the Bergman projection. For a function \( \phi \in L^\infty(\mathbb{D}) \), we define the Toeplitz operator 
\[ T^a_\phi : A^2_a(\mathbb{D}) \rightarrow A^2_a(\mathbb{D}) \]
by
\[ T^a_\phi f(z) := P_a(\phi f) \]
for \( z \in \mathbb{D} \). We say that \( \phi \) is the symbol of \( T^a_\phi \). When \( \alpha = 0 \), we will simply write \( T_\phi \). It is natural to try to study properties of \( T^a_\phi \) in terms of not only the operator itself, but its symbol.

Finding necessary and sufficient conditions for the invertibility of Toeplitz operators on spaces of holomorphic functions is an important problem in Operator Theory. For the Bergman space, Luecking found several necessary and sufficient conditions for the invertibility of Toeplitz operators with bounded, nonnegative symbols [8]. Faour then gave a necessary condition for \( T_\phi \) to be an invertible operator provided that \( \phi \) is continuous on the closed unit disc satisfying the property that \( |\phi(z_1)| \geq |\phi(z_2)| \) whenever \( |z_1| \leq |z_2| \) [5]. Using the Berezin transform, Karaev then found a sufficient condition for the invertibility of a bounded linear operator on the Bergman space [7]. Using this result, Gürdal and Şohret found a sufficient condition for a Toeplitz operator with a bounded symbol to be invertible on the Bergman space [6].

We define the Hardy space of the unit disc \( H^2(\mathbb{D}) \) to be the set of all functions \( f \) holomorphic on \( \mathbb{D} \) satisfying
\[
\sup_{0 < r < 1} \frac{1}{2\pi} \left( \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right)^{\frac{1}{2}} < \infty.
\]
Since these suprema are in fact radial limits that must exist almost everywhere on the unit circle, we may also think of the Hardy space as a space of analytic functions on the unit circle in a natural way, denoted by \( H^2 \) or \( H^2(\partial \mathbb{D}) \). The invertibility of Toeplitz operators on the Hardy space was completely characterized by Devinatz, but this characterization is not in terms of fundamental geometric or analytic properties of the symbol of the operator [2]. In an effort to construct a characterization more along these lines, Douglas showed, for continuous \( \phi \), that the Poisson extension \( |\tilde{\phi}(z)| \geq \delta > 0 \) for all \( z \in \mathbb{D} \) implies the invertibility of \( T_\phi \) [3]. He then asked the following question:

**Question 1.** If \( \phi \in L^\infty(\mathbb{T}) \) and \( |\tilde{\phi}(z)| \geq \delta \) for all \( z \in \mathbb{D} \) for some \( \delta > 0 \), is \( T_\phi \) invertible on the Hardy space?

In response, Tolokonnikov found a sufficient condition for a \( T_\phi \) to be invertible on the Hardy space [13]. Additionally, he showed that if there was a constant \( \delta > 0 \) close enough to 1 satisfying \( \delta \leq |\tilde{\phi}(z)| \leq 1 \) for all \( z \in \mathbb{D} \), that \( T_\phi \) must be invertible. Nikolski then improved this result and even obtained an estimate for the norm of the inverse operator [11]. Ultimately, Wolff negatively answered the question by constructing a highly nontrivial counterexample using martingales [15], but many authors became motivated...
to study a parallel problem on the Bergman space using a fundamental object within operator theory, the Berezin transform $\tilde{\phi}_\alpha$, defined by

$$\tilde{\phi}_\alpha(a) := \int_D \phi(z) |k_\alpha^*(z)|^2 dA(z)$$

for all $a \in \mathbb{D}$. We again omit $\alpha$ in the case $\alpha = 0$. Since the Berezin transform coincides with the Poisson extension on the Hardy space, the natural Douglas question for the Bergman space is as follows:

**Question 2.** If $\phi \in L^\infty(\mathbb{D})$ and $|\phi(z)| \geq \delta$ for all $z \in \mathbb{D}$ for some $\delta > 0$, is $T_\phi$ invertible on the Bergman space?

McDonald and Sundberg obtained positive results in the cases that $\phi$ is real harmonic, analytic or co-analytic [10]. Recently, Zhao and Zheng answered the question positively for nonnegative symbols using Luecking’s results, but also gave a counterexample to Question 2 [16]. However, the symbol provided in this counterexample is not harmonic, so the answer to Question 2 when $\phi$ is complex-valued and harmonic is still unknown.

If $\mu$ is a positive, finite, Borel measure on $\mathbb{D}$, it is possible to define the Toeplitz operator $T^\alpha_\mu : H^\infty(\mathbb{D}) \rightarrow H(\mathbb{D})$ as follows:

$$T^\alpha_\mu f(z) := \int_D K^\alpha_\mu(z) f(w) d\mu(w).$$

Using a simple density argument, it can be shown that these operators can be extended to the entire Bergman space. We say a finite, positive Borel measure $\mu$ is a Carleson measure for $A^2_\alpha(\mathbb{D})$ if there exists a constant $C > 0$ such that

$$\int_D |f|^2 d\mu \leq C \int_D |f|^2 dA(z)$$

for all $f \in A^2_\alpha(\mathbb{D})$. It turns out that $T^\alpha_\mu$ is bounded if and only if $\mu$ is a Carleson measure for $A^2_\alpha(\mathbb{D})$. When $T^\alpha_\mu$ is bounded, it is possible to define the Berezin transform $\tilde{\mu}_\alpha$ as follows:

$$\tilde{\mu}_\alpha(a) := \int_D |k_\alpha^*(z)|^2 d\mu(z).$$

Foundational information on Toeplitz operators, Carleson measures, the Berezin transform, and spaces of analytic functions can be found in [17] and [4]. In order to further pursue the problem, my advisor, Professor Željko Ćučković, and I extended Zhao and Zheng’s result to the weighted Bergman space of the unit ball in $\mathbb{C}^n$. Additionally, we asked the following natural question as a step in a new direction continuing the work above:

**Question 3.** If $\mu$ is a Carleson measure for the Bergman space and $\tilde{\mu} \geq \delta$ on $\mathbb{D}$ for some $\delta > 0$, what can be said about the Reverse Carleson properties of $\mu$?
If $\phi$ is a function in $L^\infty(\mathbb{D})$, define the averaging function $\hat{\phi}_{r,a}$ of $\phi$ by
\[
\hat{\phi}_{r,a}(a) := \frac{1}{A(a,D(a,r))} \int_{D(a,r)} \phi(z) dA(z)
\]
for each $a \in \mathbb{D}$, where $D(a,r)$ is the pseudohyperbolic disc centered at $a$ with radius $r$. Similarly, for a positive, finite Borel measure $\mu$, we define the averaging function $\widehat{\mu}_{r,a}$ by
\[
\widehat{\mu}_{r,a}(a) := \frac{\mu(D(a,r))}{A(a,D(a,r))}
\]
given $r \in (0,1)$ and for $a \in \mathbb{D}$.

As mentioned above, Zhao and Zheng’s recent paper provides an example of a $\phi \in L^\infty(\mathbb{D})$ such that $|\hat{\phi}|$ is invertible in $L^\infty(\mathbb{D})$, but whose associated Toeplitz operator is not invertible on $A^2_{\alpha}(\mathbb{D})$. Since, for such $\phi$, measures of the form $\phi dA$ are Carleson measures, this implies that the invertibility of the Berezin transform of a Carleson measure $\mu$ does not guarantee that $T^\alpha_\mu$ is invertible on the Bergman space. However, we proved the following theorem using techniques from [9] and [12]:

**Theorem 1.** [1] Let $\alpha > -1$, $0 < r < 1$, and let $\mu$ be a Carleson measure for $A^2_{\alpha}(\mathbb{D})$. If there exists $\delta > 0$ such that $\hat{\mu}_a(a) \geq \delta$ for all $a \in \mathbb{D}$, then there exists $\delta_r > 0$ such that $\hat{\mu}_{r,a}(a) \geq \delta_r$ for all $a \in \mathbb{D}$ and for $r$ sufficiently close to 1. This implies that $\mu$ is an almost Reverse Carleson measure for $A^2_{\alpha}(\mathbb{D})$. Furthermore, $\mu$ is an almost Reverse Carleson measure for $A^2_{\alpha}(\mathbb{D})$ if and only if $T^{\alpha}_{\mu}$ is invertible on $A^2_{\alpha}(\mathbb{D})$ for all $\gamma \in (0,1)$.

We obtained several partial results in the converse direction as well. Using different techniques, we were able to solve the analogous problem on the Bargmann-Fock space $\mathcal{F}^2$, defined as the closed subspace of $L^2(\mathbb{C})$ given by the set of all entire functions $f$ satisfying $\int_{\mathbb{C}} |f(z)|^2 e^{-\frac{|z|^2}{2}} dA(z) < \infty$. Much of the foundational theory on $\mathcal{F}^2$ is analogous to the theory constructed above on the Bergman space; we omit it here. Our main theorem on $\mathcal{F}^2$ is as follows:

**Theorem 2.** Suppose $\nu$ is a Fock-Carleson measure on $\mathcal{F}^2$, and suppose there exists $\delta > 0$ such that $\hat{\nu}^\beta(a) \geq \delta$ for all $a \in \mathbb{C}$. Then there exists $R_0 > 0$ and such that $\hat{\nu}^\beta_R(a) \geq \delta_R > 0$ for all $R \geq R_0$ and for all $a \in \mathbb{C}$. This implies $\nu$ is an almost Reverse-Fock Carleson measure for $\mathcal{F}^2$. Furthermore, $\nu$ is an almost Reverse-Fock Carleson measure for $\mathcal{F}^2$ if and only if the Toeplitz operator $T^{\beta}_{\nu}$ is invertible on $\mathcal{F}^2$ for all $\beta > 0$. 
Additionally, we looked at the same problem on certain Model subspaces of the Hardy space. We say a bounded, holomorphic function $u : \mathbb{D} \to \mathbb{C}$ is inner if $|u| = 1$ almost everywhere on $\partial \mathbb{D}$. Given an inner function $u$, define the Model space $\mathcal{K}_u$ by $\mathcal{K}_u := H^2 \ominus uH^2$. It is known that $\mathcal{K}_u$ is a closed subspace of the Hardy space, and is thus a reproducing kernel Hilbert space with kernel $K^u_\lambda(z) = \frac{1 - \overline{u(\lambda)}u(z)}{1 - \overline{\lambda}z}$ for $\lambda \in \mathbb{D}$ and $z \in \partial \mathbb{D}$. On these spaces, we define the Berezin transform $\widetilde{\phi}_u$ of $\phi \in L^\infty(\partial \mathbb{D})$ by

$$\widetilde{\phi}_u(\lambda) := \langle T^u_\phi k^u_\lambda, k^u_\lambda \rangle = \int_{\partial \mathbb{D}} \phi(e^{i\theta})|k^u_\lambda(e^{i\theta})|^2 d\theta,$$

for $\lambda \in \mathbb{D}$. We were able to prove the following two theorems:

**Theorem 3.** Let $\phi \in L^\infty(\partial \mathbb{D})$ be a real valued function. If there exists $\delta > 0$ such that $\widetilde{\phi}_\ast > \delta$ on $\mathbb{D}$, then $T^\ast_\phi$ is invertible on $\mathcal{K}_\ast$. However, the converse is not true.

**Theorem 4.** Let $\phi(\theta)$ be the real valued, bounded function $1 + 2\cos(2\theta)$ for $\theta \in [0, 2\pi)$. Then $\widetilde{\phi}_3(\lambda) > \frac{1}{3}$ for all $\lambda \in \mathbb{D}$, but $T^\ast_3$ is not invertible on $\mathcal{K}_3$.

### 4. Future Research

I plan to progress towards solving Douglas’ problem on the Bergman space for complex-valued harmonic symbols, or provide a counterexample. If Douglas’ problem has a positive answer there, this problem will be solved in stages by first looking at certain classes of harmonic polynomials. Since the Bergman space is a more unwieldy space than the Hardy space, however, I expect that the answer will be negative. In Zhao and Zheng’s paper mentioned above, the authors used properties of Fredholm operators and differential equations to lay some basic groundwork in these areas, and I plan to build on that infrastructure.

Additionally, I hope to solve the remainder of the question posed in my thesis:

**Question 4.** If $\mu$ is a Carleson measure for the Bergman space, and $T_{\mu_\alpha, \gamma}$ is invertible for all $\gamma > 0$, what are the minimal extra conditions $\mu$ must satisfy to guarantee that $\widetilde{\mu}_\alpha$ is invertible in $L^\infty(\mathbb{D})$?

I have made significant progress toward answering this question on both the weighted Bergman space and the Fock space.

Several fundamental Bergman space theorems regarding the invertibility of Toeplitz operators via Berezin transforms, such as Luecking’s criteria, have been shown to be valid on $\mathcal{F}^2$ [14]; I will use these results to build a system of forward and reverse Carleson measure inequalities on $\mathcal{F}^2$ that runs parallel to the theory on the Bergman space [9]. I will also further build analogous theory for Model spaces of $H^2$, this time answering Douglas’ question for Toeplitz operators with complex-valued, bounded symbols instead of only considering the case where the symbols are real-valued. I will also move beyond monomial inner function Model spaces and study the case where the inner function generating the Model space is a finite dimensional Blaschke product. Since every finite dimensional Model space is generated by a monomial or finite dimensional Blaschke...
product, this would solve Douglas’ question on all nontrivial, finite dimensional Model subspaces of the Hardy space.

I would also like to translate all of these constructions to higher dimensions. I hope to solve the same questions on the Bergman space of the bi-disk and the unit ball in $\mathbb{C}^n$, even when the Lebesgue measure is acted on by various weights. From here, I will consider Bergman spaces of Reinhardt domains and strongly pseudoconvex domains.

My colleagues have also worked on other problems Operator Theory such as invariant subspaces, hyponormality, positivity, and compactness of Toeplitz and Hankel operators on a variety of spaces and in various dimensions. I plan to collaborate with them to contribute research in these areas.

REFERENCES


