Department of Mathematics
The University of Toledo

Ph.D. Qualifying Examination

Probability and Statistical Theory

April 21, 2018

Instructions

Do all four problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
This is a closed book examination.
This is a three hour test.

1. Let $\{X_n, n \geq 1\}$ be a sequence of random variables. Show $X_n \xrightarrow{P} 0$ if and only if

$$\mathrm{E}\left(\frac{X_n^2}{1+X_n^2}\right) \to 0.$$

2. $X_1, \cdots, X_n \overset{iid}{\sim} \mathsf{Poisson}(\lambda)$.

$$f_X(x) = \frac{\lambda^x}{x!}e^{-\lambda}, \qquad x = 0, 1, \cdots$$

   a. Define $Y_i = I_{\{0\}}(X_i)$ and $T_{1n} = \sum_{i=1}^n Y_i/n$, where $I_{\{0\}}(x)$ is the indicator function and $I_{\{0\}}(x) = 1$ if $x = 0$ and 0 otherwise. Find the distribution of X such that $\sqrt{n}(T_{1n} - e^{-\lambda})$ converges to X in distribution, denoted by $\sqrt{n}(T_{1n} - e^{-\lambda}) \xrightarrow{D} X$.

   b. Find the maximum likelihood estimator T_{2n} for $e^{-\lambda}$.

   c. Find the distribution of Y such that $\sqrt{n}(T_{2n} - e^{-\lambda}) \xrightarrow{D} Y$.

   d. Both T_{1n} and T_{2n} can be used to estimate $e^{-\lambda}$. Based on the answers in (a)-(c), which of the two estimators is better? Explain.

3. [25 points] Let $\{X_n : n \geq 1\}$ be independent and exponentially distributed with mean $E(X_n) = 1$ for $n \geq 1$. Show that

$$P\left(\limsup_{n\to\infty} \frac{X_n}{\log n} = 1\right) = 1.$$

4. [25 points] Suppose $X_1, \ldots, X_n$ are independent with $X_i \sim N(\alpha + \beta t_i, 1)$, $i = 1, \ldots, n$, where $t_1, \ldots, t_n$ are known constants and α, β are unknown parameters.

(a) Find the Fisher information matrix $I(\alpha, \beta)$.

(b) Give a lower bound for the variance of an unbiased estimator of α.

(c) Suppose we know the value of β. Give a lower bound for the variance of an unbiased estimator of α in this case.

(d) Compare the estimators in parts (b) and (c). When are the bounds the same? If the bounds are different, which is larger?

(e) Give a lower bound for the variance of an unbiased estimator of the product $\alpha\beta$.

(f) Let $\alpha = 0$. Assume the parameter β is distributed as $N(0, 1)$. Find the Bayes estimate of β under squared error loss.