Ph.D. Qualifying Exam: Real Analysis

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Examiners: Z. Čučković, D. A. White

Instructions: Do 6 problems of 8. If you attempt more than 6, indicate which are to be graded.

1. Let $\phi \in L^\infty(\mathbb{R})$. (The measure on $\mathbb{R}$ is Lebesgue measure.) Show that
$$\lim_{n\to\infty} \left(\int_{\mathbb{R}} \frac{|\phi(x)|^n}{1+x^2}\, dx \right)^{1/n} = \|\phi\|_\infty$$

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $\lim_{x\to\infty} f'(x) = A$ exists. Show that
$$\lim_{x\to\infty} \frac{f(x)}{x} = A$$

3. Let $K : [0,1] \times [0,1] \to \mathbb{R}$ be continuous. If $f \in L^1(0,1)$, set
$$(Tf)(x) = \int_0^1 K(x,y) f(y)\, dy,$$
for all $x \in [0,1]$. (a) Show $Tf \in C([0,1])$. (b) Let B be the unit ball of $L^1(0,1)$ and show that $T(B)$ is relatively compact in $C([0,1])$.

4. In $C[0,1]$, let
$$\mathcal{A} = \text{span}\{x^n(1-x) : n \geq 1\}.$$
Prove that $\mathcal{A}$ is an algebra whose uniform closure is $\{f \in C[0,1] : f(0) = f(1) = 0\}$.

5. Suppose that $f \in L^1(\mathbb{R})$ and A is a Borel subset of $\mathbb{R}$. Show that the mapping
$$t \mapsto \int \chi_{A+t} f(x)\, dx$$
is continuous from $\mathbb{R}$ to itself. Here $A+t = \{x+t : x \in A\}$. (Suggestion: Begin with the case that A is an interval.)

6. Suppose that f_n, $n \in \mathbb{N}$ is a sequence of functions defined on $A \subseteq \mathbb{R}$ and uniformly convergent to a function f. Assume x_0 is a limit point of A and $\lim_{x\to x_0} f_n(x)$ exists for all $n \in \mathbb{N}$. Prove that

$$\lim_{n\to\infty} \lim_{x\to x_0} f_n(x) = \lim_{x\to x_0} f(x)$$

(in the sense that one limit exists if and only if the other does and they are equal).

7. Let f be a non-negative element of $L^1(0,\infty)$ and let A be a Borel subset of $(0,\infty)$.

   (a) Suppose the Lebesgue measure $m(A)$ of A is finite. Prove that

   $$\lim_{n\to\infty} \int_A (f(x))^{1/n} dx = m(\{x \in A : f(x) > 0\})$$

   (b) Consider the case when $m(A) = \infty$. What can be said about

   $$\lim_{n\to\infty} \int_A (f(x))^{1/n} dx$$

   in this case?

8. Suppose that $f \in L^1(\mathbb{R})$. Show that, for almost all x.

$$\lim_{h\to 0} \frac{1}{h} \int_{|y-x|<h} |f(y) - f(x)| dy = 0$$

Suggestion: Begin with the case that f is continuous and of compact support.