Ph.D. Qualifying Exam

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Instructions: Do six of the 9 questions. No materials are allowed.

1. (a) State the Arzela Ascoli Theorem.
   (b) Consider the set $K = \{f \in C[0,1] : |f(x)| \leq x(1-x), 0 \leq x \leq 1\}$. Determine whether or not K is compact in $C[0,1]$. Show your reasoning.

2. Consider $f(x) = \sum_{n=0}^{\infty} \frac{1}{1+n^2 x}$, defined for $0 < x \leq 1$. Show that the series defining f converges uniformly on compact subsets of the interval $0 < x \leq 1$. Show further that f is unbounded. Finally determine whether or not $f \in L^1(m)$ if m is Lebesgue measure on the interval $0 < x \leq 1$.

3. (a) Show that $L^p(-1,1) \subset L^q(-1,1)$ if $\infty > p > q \geq 1$.
   (b) Show that $L^p(-1,1) \neq L^q(-1,1)$ if $\infty > p > q \geq 1$.
   (c) Show that $\cap_p L^p(-1,1) \supsetneq L^\infty(-1,1)$.

4. Suppose $(\Omega, \mathcal{F}, m)$ is a σ-finite measure space and $f \in L^1(m)$ is non-negative. Define

$$\mu(A) = \int_A f\, dm, \quad \text{for all } A \in \mathcal{F}$$

   (a) Show that μ is a measure.
   (b) Show that , for any $h \in L^\infty(m)$, $\int_\Omega h\, d\mu = \int_\Omega hf\, dm.$
   (c) Suppose the $g \in L^1(\mu)$ and g is nonnegative and that ν is defined by $\nu(A) = \int_A g\, d\mu$, for all $A \in \mathcal{F}$. Show that $fg \in L^1(m)$ and

$$\nu(A) = \int_A fg\, dm, \quad \text{for all } A \in \mathcal{F}.$$

5. (a) Suppose that f is continuously differentiable on a compact interval $[a, b]$. Show that f is of bounded variation and

$$\text{Var}_{[a,b]} f = \int_a^b |f'(x)|\, dx$$

Recall that $\text{Var}_{[a,b]} f = \sup\{\sum_{1\leq j\leq m} |f(x_j) - f(x_{j-1})|$ where the supremum is taken over all partitions, $a = x_0 < x_1 < \ldots < x_m = b$ of $[a, b]$.

(b) Show that

$$f(x) = \begin{cases} x^\alpha \sin\left(\frac{1}{x}\right) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

is of bounded variation on [0,1] if $\alpha > 1$.

6. (a) Let $\{s_n\}$ be a sequence os real numbers and

$$\sigma_n = \frac{s_1 + s_2 + \ldots + s_n}{n}.$$

Show that if $\lim_{n\to\infty} s_n$ exists, then $\lim_{n\to\infty} \sigma_n$ exists and both limits are equal.

(b) Further show that if $\lim_{n\to\infty} \sigma_n$ exists and $\lim_{n\to\infty} ns_n = 0$, then $\lim_{n\to\infty} s_n$ exists.

7. (a) If f is a real-valued function defined on the interval (a, b) and satisfies

$$f(tx + (1-t)y) \leq f(x) + (1-t)f(y)$$

whenever $0 < t < 1, a < x, y < b$, we call f a convex function. Show that any convex function is continuous.

(b) On the other hand if f is continuous and satisfies

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}$$

for all $a < x, y < b$, show that f is convex.

8. Put $P_0 = 0$. Define for $n = 0, 1, 2, \ldots$,

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}.$$

Prove that $\lim_{n\to\infty} P_n(x) = |x|$ uniformly on $[-1, 1]$.[Hint: Use the identity

$$|x| - P_{n+1}(x) = [|x| - P_n(x)]\left[1 - \frac{|x| + P_n(x)}{2}\right]$$

to prove that

$$0 \le P_n(x) \le P_{n+1}(x) \le |x|$$

if $|x| \le 1$ and that

$$|x| - P_n(x) \le |x|\left(1 - \frac{|x|}{2}\right)^n < \frac{2}{n+1}$$

if $|x| \le 1$.

9. (a) Show that if $f(t)$ is continuous on the real line and $0 < a < b$ then the integral $\displaystyle\int_a^b \frac{f(t)}{t}dt = \frac{1}{a}\int_a^s f(t)dt$ for some s in the interval $[a, b]$.

(b) Show that the integrals $\displaystyle\int_0^\infty \sin(t^2)dt$, $\displaystyle\int_0^\infty \cos(t^2)dt$ are conditionally convergent.