Topology Ph.D. Qualifying Exam

Gerard Thompson   Mao-Pei Tsui

April 5 2008

This examination has been checked carefully for errors. If you find what you believe to be an error in a question, report this to the proctor. If the proctor's interpretation still seems unsatisfactory to you, you may modify the question so that in your view it is correctly stated, but not in such a way that it becomes trivial. If you feel that the examination is on the long side do not panic. The grading will be adjusted accordingly.

1 Part One: Do six questions

1. If (X, d) is a metric space then $\{x \in X : d(x, x_0) \leq \epsilon\}$ is said to be the closed ball of radius ϵ and center x_0. Prove that a closed ball is a closed set.

2. Define what it means for (X, d) to be a metric space. Then $d : X \times X \to \mathbb{R}$: is d continuous? Discuss. If you cannot answer in general do it for $X = \mathbb{R}$ with the usual topology.

3. Prove or disprove: if a metric space is compact then it is bounded.

4. A set is said to have the finite complement topology if the closed sets are the finite sets together with the empty set. Let $f : \mathbb{R} \to \mathbb{R}$ be the identity map $f(x) = x$ where in the domain $\mathbb{R}$ has the usual topology but in the codomain it has the finite complement topology. Show that f is continuous. Is f a homeomorphism? Explain your answer.

5. Define the term *closure* $\overline{A}$ of a subspace A of a topological space X. Prove that if A and B are subspaces of X and B is closed and $A \subset B$ then $\overline{A} \subset B$.

6. Let $X = \Pi_{\mu \in M} X_\mu$ be the Cartesian product of the topological spaces $(X_\mu)_{\mu \in M}$ and let X have the product topology. Recall that a space is T_1 if any two distinct points in the space can be separated by not necessarily disjoint open sets. Show that if each $(X_\mu)_{\mu \in M}$ is T_1 then so is X. If you cannot do it for T_1 spaces do it for Hausdorff spaces.

7. Define compactness for a topological space. A collection of subsets is said to have the finite intersection property if every finite class of those subsets has non-empty intersection. Prove that a topological space is compact iff every collection of closed subsets that has the finite intersection property itself has non-empty intersection.

8. It is a fact that every compact subset of a Hausdorff space is closed. Moreover a topological space is said to be *normal* if every pair of disjoint closed sets can be separated by disjoint open sets. Prove that a compact Hausdorff space is normal.

9. Let X be a topological space. Let $A \subset$ X be connected. Prove that the closure $\overline{A}$ of A is connected.

10. Define the term "identification map." If f maps open sets to open sets and is surjective show that f is an identification map. What happens if we replace "open" by "closed" in the preceding sentence?

11. Prove or disprove: in a compact topological space every infinite set has a limit point. If you cannot answer the question for a compact topological space answer it for a metric space.

12. Prove that $\mathbb{R}$ is connected.

2 Part Two: Do three questions

1. Let S^1 denote the unit circle in $\mathbb{R}^2$. Define $f : S^1 \to S^1$ by $f(x) = -x$. Prove that f is homotopic to the identity map. Now suppose that $g : S^1 \to S^1$ is a map that is not homotopic to the identity map. Show that $g(x) = -x$ for some $x \in S^1$.

2. Let $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ be the closed unit disk. Prove that D in cannot be retracted to the unit circle S^1. Deduce that any continuous map $f : D \longrightarrow D$ has a fixed point. (*Hint:* Consider the line joining x to $f(x)$ where $x \in D$.)

3. (i) Prove that $\pi_1(X\times Y)$ is isomorphic to $\pi_1(X)\times\pi_1(Y)$ where X and Y are connected topological spaces.

   (ii) Is the map $f : (0, 13) \to S^1$ defined by $f(x) = e^{2\pi ix}$ a covering map? Explain.

4. (i) The polygonal symbol of a certain surface without boundary is $xyzyzx$. Identify the surface. What is its Euler characteristic?

   (ii) Explain how polygons with an even number of sides may be used to classify surfaces without boundary. You do not need to give detailed proofs.

5. (i) Prove that a retraction induces an epimorphism of fundamental groups.

   (ii) Compute the fundamental group of the real projective plane by any valid method.

6. State the simple version of the simplicial approximation theorem. Let a regular hexagon with vertices $ABCDEF$ be inscribed in a circle and consider the equilateral triangle ACE. Let K and L denote the two-dimensional simplicial complexes consisting of the two-simplexes ABC, CDE, EFA, ACE together with all their faces and let L denote the two-dimensional simplicial complexes consisting of the two-simplex ABC together with all its faces. Define a map $f : |K| \to |L|$ by projecting towards the center of the circle. Describe Star(A,K) (the open star) and find a simplicial approximation to f. Is there a simplicial approximation to f^{-1}? Explain.

7. For the sake of this problem a manifold of dimension n will be defined as a topological space in which each point has a neighborhood that is homeomorphic to $\mathbb{R}^n$. If M is a connected manifold of dimension at least 3 and $q \in M$, show that $\pi_1(M - \{q\})$ is isomorphic $\pi_1(M)$.