Ph.D. Qualifying Exam: Real Analysis

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Instructions: Do six of the 9 questions. No materials are allowed.
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1. Suppose that \((\Omega, \mathcal{F}, \mu)\) is a finite measure space and that \(f_n\) is a sequence in \(L^1(\Omega, \mathcal{F}, \mu)\) which converges to 0 in \(L^1(\Omega, \mathcal{F}, \mu)\).

   (a) Give an example to show that \(f_n\) need not converge to 0 almost everywhere.

   (b) Show that \(f_n\) converges in measure to 0.

   (c) Suppose that some subsequence of the \(f_n\) converges pointwise almost everywhere to some function \(f\). Must \(f = 0\) almost everywhere? Explain.

2. Suppose \(f\) and \(g\) are nonnegative integrable functions defined on a measure space \((\Omega, \mathcal{F}, \mu)\).

   (a) Show that \(\min\{\int f\,d\mu, \int g\,d\mu\} \geq \int \min\{f, g\}\,d\mu\).

   (b) If equality holds then what can be said about the relationship between \(f\) and \(g\)?

3. (a) Give an example of a sequence of bounded functions which are Riemann integrable on a compact interval \([a, b]\) and the sequence converges pointwise to a function which is not Riemann integrable.

   (b) Give an example of a function \(f\) which is not Lebesgue measurable on \([a, b]\) but \(f^2\) is.

   (c) Give an example of a function \(f\) which is Lebesgue integrable on \([a, b]\) but \(f^2\) is not.

4. (a) State the Baire Category theorem. If you use the terminology “first category” or “second category” then you should define those terms.
5. Prove or disprove.

(a) Every absolutely continuous function defined on $[0,1]$ is of bounded variation.

(b) Every continuous function defined on $[0,1]$ is of bounded variation.

(c) If $f$ is continuous and increasing on $[0,1]$ then $f(1) - f(0) = \int_0^1 f'(x) \, dx$.

6. Suppose that $(\Omega, \mathcal{F}, \mu)$ is a measure space and $f_n$ is a sequence of real valued Borel measurable functions $f_n : \Omega \to \mathbb{R}$ such that

$$\sum_{n \in \mathbb{N}} \int_{\Omega} |f_n| \, d\mu < \infty$$

Show that $\sum_n f_n(x)$ converges $\mu$-almost everywhere to a function $f(x)$ say and $f \in L^1(\mu)$ and

$$\int_{\Omega} f \, d\mu = \sum_{n \in \mathbb{N}} \int_{\Omega} f_n \, d\mu$$

7. Consider the sequence $f_n(x) = e^{-n\sqrt{x}}$. Show that, for any $a > 0$ $f_n$ converges to 0 uniformly on $[a, \infty)$ but $f_n$ does not converge uniformly on $(0, \infty)$. Compute

$$\lim_{n \to \infty} \int_0^\infty f_n(x) \, dx$$

and explain your answer.

8. Let $I = [0,1]$ and $K = I^n$. Fix $\alpha$ such that $0 < \alpha < 1$. Let $S$ be the family of all real valued functions on $K$ for which

$$\|f\|_\alpha = \left( \sup_K |f(x)| + \sup_{K \times K} \frac{|f(x) - f(y)|}{|x - y|^\alpha} \right) \leq 1.$$

Show that the closure of $S$ in $C(K)$, the space of continuous functions on $K$ with the supremum norm, is compact.
9. Let $f$ be a continuous function on $[0, \infty)$ and \( \int_0^\infty |f(x)| \, dx < \infty \). Assume that \( \int_0^\infty f(x) e^{-nx} \, dx = 0 \) for all integers $n$ sufficiently large. Show that $f \equiv 0$. 