Topology Ph.D. Qualifying Exam

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This examination has been checked carefully for errors. If you find what you believe to be an error in a question, report this to the proctor. If the proctor's interpretation still seems unsatisfactory to you, you may modify the question so that in your view it is correctly stated, but not in such a way that it becomes trivial. If you feel that the examination is on the long side do not panic. The grading will be adjusted accordingly.

1 Part One: Do six questions

1. If (X, d) is a metric space then $\{x \in X : d(x, x_0) < \epsilon\}$ is said to be the open ball of radius ϵ. Prove that an open ball is an open set.

2. A set is said to have the finite complement topology if the closed sets are the finite sets together with the empty set. If X is a topological space with an infinite number of points show that the diagonal $\Delta := \{(x, x) | x \in X\}$ is not closed in the finite complement topology.

3. A topological space X is said to be *locally connected* if any neighborhood of any point contains a connected neighborhood. Prove that the connected components of a locally connected space are both open and closed.

4. Let D be the open unit disk in the complex plane that is $D := \{z| \ |z| < 1\}$. Let $\sim$ be an equivalence relation on D defined by $z_1 \sim z_2$ if $|z_1| = |z_2|$. Is the quotient space $D/\sim$ Hausdorff? Prove or disprove.

5. Let B be an open subset of a topological space X. Prove that a subset $A \subset B$ is relatively open in B if and only if A is open in X.

6. Define compactness for a topological space. Prove from your definition that the closed interval $[0, 1]$ is compact.

7. Let $X = \Pi_{\mu\in M} X_\mu$ and $Y = \Pi_{\mu\in M} Y_\mu$ be the Cartesian products of the topological spaces $(X_\mu)_{\mu\in M}$ and $(Y_\mu)_{\mu\in M}$ and let X and Y have the product topologies, respectively. Prove that if for each $\mu \in M$ the maps $f_\mu : X_\mu \to Y_\mu$ are continuous then $f : X \to Y$ defined by $f(x)_\mu = f_\mu(x_\mu)$ is continuous.

8. Prove that if two connected sets A and B in a space X have a common point p, then $A \cup B$ is connected.

9. Prove that every closed subspace of a locally compact space is locally compact.

10. It is a fact that every compact subset of a Hausdorff space is closed. Moreover a topological space is said to be *normal* if every pair of disjoint closed sets can be separated by disjoint open sets. Prove that a compact Hausdorff space is normal.

11. Let X be a topological space. Let $A \subset$ X be connected. Prove that the closure $\overline{A}$ of A is connected.

12. Prove that if $f : X \mapsto Y$ is continuous and surjective and X is compact and Y Hausdorff then f is an identification map.

2 Part Two: Do three questions

1. Recall that $S^k = \left\{(x_1, \cdots, x_{k+1}) \middle| x_1^2 + \cdots x_{k+1}^2 = 1\right\} \subset \mathbb{R}^{k+1}$. The antipodal map $A_k : S^k \mapsto S^k$ is the smooth map defined by $(x_1, \cdots, x_{k+1}) \mapsto (-x_1, \cdots, -x_{k+1})$.

   (i) Show that $A_1 : S^1 \mapsto S^1$ is homotopic to the identity map.

   (ii) Show that $A_k : S^k \mapsto S^k$ is homotopic to the identity map if k is odd.

2. (i) Let X be a topological space. Let $f, g : I \mapsto X$ be two paths from p to q. Show that $f \sim g$, that is, f is homotopic to g if and only if $f \cdot g^{-1} \sim c_p$ where g^{-1} is the inverse path to g, c_p denotes the constant path based at p and the " $\cdot$ " denotes the product of (compatible) paths.

   (ii) Show that X is simply connected if and only if any two paths in X with the same initial and terminal points are path homotopic.

3. (i) The polygonal symbol of a certain surface without boundary is $xyzyzx$. Identify the surface. What is its Euler characteristic?

   (ii) Explain how polygons with an even number of sides may be used to classify surfaces without boundary. You do not need to give detailed proofs.

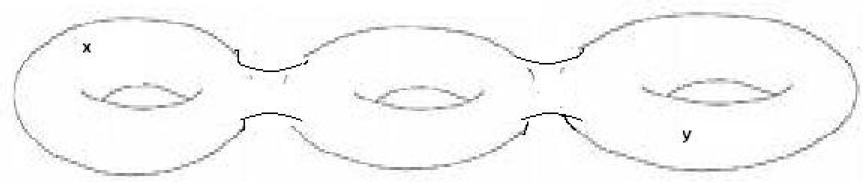
4. Let $X = [0, 1]\times[0, 1]$ denote the rectangle in $\mathbb{R}^2$. Let $\sim$ be the equivalent relation generated by $(0, p) \sim (1, 1 - p)$ wher $0 \le p \le 1$. The quotient space $X/\sim$ is called the Möbius band. Show that S^1 is a retract of the Möbius band.

5. Compute the first three homology groups of the *hollow* torus $S^1 \times S^1$. You may use simplicial theory and a triangulation but do not simply say that $H_1(S^1 \times S^1)$ is the abelianization of $\pi_1(S^1 \times S^1)$.

6. Let $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ be the closed unit disk. Prove that D in cannot be retracted to the unit circle S^1. Deduce that any continuous map $f : D \longrightarrow D$ has a fixed point. (*Hint:* Consider the line joining x to $f(x)$ where $x \in D$.)

7. (i) Compute the fundamental group of the *solid* torus in the figure below when the points x and y are identified.

   (ii) Compute the fundamental group of the *hollow* torus in the figure below when the points x and y are identified.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