TOPOLOGY PH.D. QUALIFYING EXAM

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This exam has been checked carefully for errors. If you find what you believe to be an error in a question, report this to the proctor. If the proctor's interpretation still seems unsatisfactory to you, you may alter the question so that in your view it is correctly stated, but not in such a way that it becomes trivial.

SECTION 1

Do 3 of the following 5 problems.

(1) Prove that $(0, 1)$ is homeomorphic to $\mathbb{R}$.

(2) Let X be a topological space which is connected and locally path connected. Prove that X is path connected.

(3) Prove that a connected space is path connected if and only if every path component is open.

(4) Prove that a space Y is Hausdorff if and only if for every space X and every pair of continuous functions $f : X \mapsto Y$ and $g : X \mapsto Y$, the set $\{x \in X | f(x) = g(x)\}$ is closed in X.

(5) Let $f : X \mapsto Y$ be a quotient map, with Y connected. Show that if $f^{-1}(y)$ is connected for all $y \in Y$, then X is connected.

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Section 2

Do 3 of the following 5 problems.

(1) Prove
  (a) Define $f, g : X \longrightarrow S^n$ are continuous and $f(x) \neq -g(x)$ for all $x \in X$, then f is homotopic to g.
  (b) A continuous $f : S^n \mapsto S^n$ either has a fixed point or is homotopic to the antipodal map.

(2) Let X be a connected, locally path-connected space. Suppose $\pi_1(X)$ is finite. Show that every continuous map $f : X \mapsto S^1$ is homotopic to a constant map.

(3) Let X_1 and X_2 be two copies of S^2 and let N_1, S_1 and N_2, S_2 be the north and south poles of X_1 and X_2, respectively. Define X to be the quotient space obtained by identifying N_1 with N_2 and S_1 with S_2. Compute the fundamental group of X by using the Seifert-van Kampen theorem.

(4) Let S^1 be the unit circle in $\mathbb{R}^2$ and $p : X \mapsto S^1$ be a covering map with finitely many sheets. Prove that X is homeomorphic to S^1.

(5) Let $p : \widetilde{X} \mapsto X$ be the universal covering space of a space X and let $f : X \mapsto X$ be a continuous map.
  (a) Prove that there exist lifts of f to $\widetilde{X}$, that is, maps $\widetilde{f} : \widetilde{X} \mapsto \widetilde{X}$ such that $p \circ \widetilde{f} = f \circ p$.
  (b) Suppose $\widetilde{f_1}$, $\widetilde{f_2}$ are lifts of f and there exists $\widetilde{x_1}$, $\widetilde{x_2}$ such that $\widetilde{f_1}(\widetilde{x_1}) = \widetilde{x_1}$, $\widetilde{f_2}(\widetilde{x_2}) = \widetilde{x_2}$ and $p(\widetilde{x_1}) = p(\widetilde{x_2})$. Prove that there exists a covering transformation $\sigma : \widetilde{X} \mapsto \widetilde{X}$ such that $\widetilde{f_2} = \sigma \widetilde{f_1} \sigma^{-1}$.