Algebra Ph.D. Qualifying Exam- April 15, 2006

Instructions: The exam is divided into three sections. Please choose exactly three problems from each section. Clearly indicate which three you would like graded. You have three hours.

$\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote, respectively, the rational numbers, the real numbers and the complex numbers.

1. Section I

(1) Classify completely the possible isomorphism type of a group with 2006 elements.

(2) Suppose $f : G \to A$ is a group homomorphism, A is abelian. Prove any subgroup of G which contains $\ker f$ is normal.

(3) If G is a group, let $D = \{(x,x) : x \in G\} \leq G \times G$. Prove that G is a simple group if and only if D is a maximal subgroup of $G \times G$.

(4) Let P be a finite p-group. Prove the center of P is nontrivial.

(5) Recall that for subgroup $H \leq G$ we have the *normalizer* of H:

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\}$$

and the *centralizer* of H:

$$C_G(H) = \{g \in G \mid gh = hg \ \forall h \in H\}.$$

Let G be a finite group and P a Sylow p-subgroup of G. Let π be the permutation representation of G acting on the left cosets of $N_G(P)$. Prove:

(a) $\pi(P)$ fixes exactly one letter (i.e. one coset).

(b) Suppose $|P| = p$ and let $x \in P$, $x \neq e$. Then $\pi(x)$ is a product of one 1-cycle and a certain number of p-cycles.

(c) If $|P| = p$ and $y \in N_G(P) - C_G(P)$, then $\pi(y)$ fixes at most r-letters where r denotes the number of orbits under the action of $\pi(P)$.

2. Section II

(6) Let I and J be ideals in a commutative ring R (with 1) and suppose that $I + J = R$.

(a) Prove that $IJ = I \cap J$.

(b) Show that, as R-modules, $I \oplus J \cong R \oplus IJ$.

(c) Give an example of two such ideals I and J such that neither is principal. [*Hint:* Consider $R = \mathbb{Z}[x]$.]

(7) Let F be a field, $F[x]$ and $F[x, y]$ polynomial rings in one and two commuting variables.

a. Prove $F[x]$ is a principal ideal domain. Determine all the maximal ideals.

b. Determine all the maximal ideals in $F[x, y]$. Is $F[x, y]$ a principal ideal domain? Explain.

(8) Let R be a commutative ring with identity. Prove that the subset of R containing 0 together with all zero divisors in R must contain at least one prime ideal.

(9) Let R be a commutative Noetherian ring with 1.

(a) If $f : R \to R$ is a surjective ring homomorphism, prove that f is an isomorphism.

(b) Show that the rings R and $R[x]$ are not isomorphic.

(c) Give an example to show that (b) can fail if R is not Noetherian.

(10) Let ϵ be a primitive n^{th} root of unity in the complex numbers. If m is an integer such that $m > 2$, show that the polynomial $x^m - 2$ has no roots in $\mathbb{Q}(\epsilon)$.

3. Section III

(11) Let $f(x) \in \mathbb{Q}[x]$ with deg $f = n$ and let K be a splitting field of $f(x)$ over $\mathbb{Q}$. Suppose that the Galois group $G(K/\mathbb{Q})$ is isomorphic to the symmetric group S_n.

(a) Show that $f(x)$ is irreducible over $\mathbb{Q}$.

(b) If $n > 2$ and α is a root of $f(x)$ in K, show that the only automorphism of $\mathbb{Q}(\alpha)$ is the identity.

(c) If $n \geq 4$, show that $\alpha^n \notin \mathbb{Q}$.

(12) Prove the multiplicative group of nonzero elements in a finite field is cyclic.

(13) a. Write down a matrix which has characteristic polynomial $c(x) = (x-1)^3(x-2)^3$ and minimal polynomial $m(x) = (x-1)^2(x-2)$.

b. Are the two matrices below similar? Justify your answer:

$$A = \begin{pmatrix} 0 & 2 & 1 \\ -1 & 3 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 1 \\ -2 & 5 & 2 \\ 2 & -4 & -1 \end{pmatrix}$$

(14) Let $A, B \in M_{n\times n}(\mathbb{C})$ such that B is invertible. Prove there exists a scalar $\alpha \in \mathbb{C}$ such that $A + \alpha B$ is not invertible.

(15) True or false All questions are for $n \times n$ matrices over $\mathbb{C}$ unless specifically stated.

a. The Jordan canonical form of a diagonal matrix is the matrix itself.

b. Matrices with the same characteristic polynomial are similar.

c. Every matrix is similar to its Jordan canonical form.

d. If a linear operator has a Jordan canonical form, then there is a unique Jordan canonical basis for that operator.

e. If the characteristic polynomial of A has no multiple roots then A is diagonalizable.

f. A matrix satisfying $A^2 = A$ must be diagonalizable.

g. An invertible matrix A is diagonalizable if and only if A^{-1} is.

h. Interchanging two columns of a matrix preserves the determinant.

i. There exists a 5×4 matrix A such that AA^τ is invertible.

j. The product of two eigenvalues of A is also an eigenvalue of A.