Differential Equations - Ph.D. Exam

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The Examination Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

Ph.D. Qualifying Examination

You should work <u>any three</u> of the four problems on each of the two parts (ODE, PDE). Show all your work and clearly indicate your answers.

PART ODE

1. Sketch the trajectory corresponding to the solution of

$$\frac{dx}{dt} = -x, \ \frac{dy}{dt} = -2y; \ x(0) = 4, \ y(0) = 2.$$

   Find all equilibrium points for the system and indicate the direction of motion on the trajectory for increasing t.

2. Consider the linear system

$$\frac{dx}{dt} = ax + by, \ \frac{dy}{dt} = cx + dy \ (a, b, c, d \text{ real}).$$

   (a) Show that if $ad - bc \neq 0$, the only critical point is $(0, 0)$.

   (b) Show that if $ad - bc = 0$, then there are an infinite number of such points. Is $(0, 0)$ an "isolated" critical point? Why?

3. Given the set of non-zero functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \cdots, \phi_m(x)\}$ which are mutually orthogonal on $a \leq x \leq b$, prove they are linearly independent.

4. Find the general solution for the system by uncoupling the system:

$$x'(t) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}.$$

PART PDE

1. The heat equation that we have been solving is linear, whether homogeneous or not. Consider the more physically reasonable model of heat conduction where the conductivity k depends on the temperature u:

$$\frac{\partial}{\partial x}\left[k(u)\frac{\partial u}{\partial x}\right] = c_v\,\frac{\partial u}{\partial t}.$$

   Suppose that $k(u) = k_0 u$ for constants k_0 and c_v. Rewrite the equation by differentiating and letting $\alpha = c_v/k_0$. If two solutions, $u_1(x,\,t)$ and $u_2(x,\,t)$, are known, is their sum a solution? Why?

2. Solve the equation for $u(x,\,y)$ by separation of variables:

$$x^2 u_{xx} + y^2 u_{yy} + 5x u_x - 5y u_y + 4u = 0.$$

3. If $w = f(x,\,y),\ x(u,\,v) = u\cosh v,\ y(u,\,v) = u\sinh v$; show that

$$w_x^2 - w_y^2 = w_u^2 - \frac{1}{u^2}\,w_v^2.$$

4. Find the function $u(x,\,t)$ satisfying the following four conditions:

$$\begin{array}{ll} u_t = u_{xx} & 0 < x < 1,\ 0 < t < \infty \\ u(0,\,t) = u(1,\,t) = 0 & 0 < t < \infty \\ u(x,\,0) = 1 & 0 \le x \le 1. \end{array}$$