Ph.D. Qualifying Examination
Spring 2002

Instructions:

1. If you think that there is a mistake ask the proctor. If the proctor's explanation is not satisfactory, interpret the problem as you see fit, but not in such a way that the answer is trivial.
2. From each part solve 3 of 5 problems.
3. If you solve more that three problems from a part indicate the problems that you wish to have graded.

Part A

1. Find the fundamental solution of the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 3x & & +4z \\ 2x & & +3z \\ -2x & +y & -2z \end{bmatrix}.$$

2. Consider the system

$$\begin{aligned} \dot{x} &= \sin(t)y \\ \dot{y} &= -\cos(t)x \end{aligned}$$

Suppose a solution $u(t) = (x(t), y(t))$ has initial values $u(0) = u_0 = (0, 1)$. Show that for $0 \leq t \leq \pi$

$$||u(t) - u_0|| \leq \frac{1}{2}(1 - \sqrt{2}\cos(t - \frac{\pi}{4})).$$

3. Consider the equation $\dot{x} = Ax + B(t)x$ where A is an $n \times n$ matrix all of whose eigenvalues have strictly negative real part and $B(t)$ is a continuous $n \times n$-matrix valued function that satisfies $||B(t)|| \leq ce^{-\beta t}$. Prove that $x(t) = 0$ is an asymptotically stable solution.

4. Suppose that $X\colon \mathbf{R}^2 \to \mathbf{R}^2$ is a Lipschitz continuous dynamical system on $\mathbf{R}^2$ with flow φ_t. Let K be a compact subset of $\mathbf{R}^2$ that contains no singular

points. Prove that if for some $p \in \mathbf{R}^2$, $\varphi_t(p)$ enters K, then it must leave K in finite time.

5. Consider the system $\dot{x} = (1-x^2)a(t)$ for a continuous function $a(t)$. What condition on $a(t)$ implies that the solution $x(t) = 1$ is Lyapunov stable. What condition implies that $x(t) = 1$ is asymptotically stable. Find an $a(t)$ so that $x(t) = 1$ is uniformly asymptotically stable.

Part B

1. Find the canonical form and the general solution of the equation

$$2xu_{xx} + 2(1+xy)u_{xy} + 2yu_{yy} + \frac{2(1-x)}{1-xy}u_x + \frac{2(1-y)}{1-xy}u_y = 0.$$

2. Consider the linear system for two vector valued functions $\mathbf{E}\colon \mathbf{R}\times\mathbf{R}^3 \to \mathbf{R}^3$ and $\mathbf{B}\colon \mathbf{R}\times\mathbf{R}^3 \to \mathbf{R}^3$. If $p \in \mathbf{R}\times\mathbf{R}^3$ is represented by $p = (t, \mathbf{x})$ then $\mathbf{E}(t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$ satisfy the linear system

$$\begin{array}{ll} \nabla\cdot\mathbf{E} = \rho & \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} & \nabla\times\mathbf{B} = \mathbf{j} + \frac{\partial\mathbf{E}}{\partial t} \end{array}$$

Describe the symbol of this equation and show that it is of hyperbolic type.

3. Let A_1 and A_2 be 2×2 matrices with $A_1 = \begin{bmatrix} A_{11} \\ A_{12} \end{bmatrix}$ and $A_2 = \begin{bmatrix} A_{21} \\ A_{22} \end{bmatrix}$ where A_{11}, A_{12}, A_{21}, and A_{22} are nonzero elements of $\mathbf{R}^2$ representing the rows of A_1 and A_2. Suppose that $u(x,y) = \begin{bmatrix} u_1(x,y) \\ u_2(x,y) \end{bmatrix}$ satisfies the system

$$A_1\frac{\partial}{\partial x}u(x,u) + A_2\frac{\partial}{\partial y}u(x,y) = 0.$$

Also suppose that (A_{11}, A_{21}) and (A_{12}, A_{22}) are independent in $\mathbf{R}^4$ and that either A_{11} and A_{21} are independent or A_{12} and A_{22} independent in $\mathbf{R}^2$. Show that by a change of variables of the form $w(x,y) = Cu(B\begin{bmatrix} x \\ y \end{bmatrix})$ this system can be placed in Cauchy-Kowalewski form. Here B and C are 2×2 matrices.

4.

(a) Give an example of a subharmonic function u on $B_R(0)$ that is not harmonic and has the boundary values $u|_{\partial B_R(0)} = \varphi$ for some continuous function φ on $\partial B_R(0)$.

(b) Let $\Omega \subset \mathbf{R}^n$ be an open subset and let u be a harmonic function on Ω. Suppose the for $x \in \Omega$, $B_R(x)$ is a open ball about x with $B_R(x) \subset \Omega$ and let v be a subharmonic function defined on $B_R(x)$ with $u|_{\partial B_R(x)} = v|_{\partial B_R(x)}$ Show that the function

$$V(x) = \begin{cases} u(x) & x \in B_R(x)^{\sim} \\ v(x) & x \in B_R(x) \end{cases}$$

is subharmonic.

5. Consider the 2-dimensional wave equation in cylindrical coordinates $u_{tt} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$ with domain the unit disk $D^1 = \{(r,\theta)|0 \le r < 1\}$ and $t > 0$ and with the boundary conditions $u(r,\theta,0) = 1 - r^2$ and $u_t(r,\theta,0) = 0$.