SECTION I

1. Let G be a finite group and let p be a prime. Suppose that every proper subgroup of G of order divisible by p is a p-group. Show that every two distinct Sylow p-subgroups of G intersect trivially. (*Hint*: Consider the normalizer of a maximal intersection of Sylow p-subgroups.)

2. Let G be a finite simple group.

   (a) If G has a proper subgroup of index less than or equal to 9, show that G has no elements of order 21.

   (b) If $|G| = 504$, show that G has no elements of order 21.

   (c) If $|G| = 504$, find the number of Sylow 7-subgroups of G. (Justify your answer.)

SECTION II

3. Let E/F be a finite dimensional field extension, and let G be the group of automorphisms of E which fix each element of F. Suppose that for some $u \in E$ the elements $\sigma(u)$ (as σ ranges over G) form a basis for E over F.

   (a) Show that if an element v of E is fixed by each σ in G, then v is in F.

   (b) Show that $E = F[u]$.

4. Let E/F be a field extension. Give a complete proof of the fact that E is an algebraic extension of F if and only if each subring of E that contains F is a field.

SECTION III

5. Let R be a primitive ring in which $a(ab - ba) = (ab - ba)a$ for all a and b in R. Show that R is a division ring.

6. A ring R is called *prime* if whenever I and J are ideals of R with $IJ = 0$, then $I = 0$ or $J = 0$. (IJ is the additive subgroup of R generated by the set of products xy with $x \in I$ and $y \in J$.)

   (a) Show that if a prime ring has no nonzero nilpotent elements then it has no nonzero zero divisors. (Any properties of prime rings that you use other than the definition must be verified.)

   (b) Show that a primitive ring is prime.

7. Let M be a finitely generated unital right R-module over the right noetherian ring R. Show that M is right noetherian.

DEPARTMENT OF MATHEMATICS
Ph.D. Comprehensive Exam
Algebra
September 1996

Instructions

Do four problems. Do at least
one problem from each section.

Please make sure that you give complete
solutions to each problem that you do.

If you attempt more than four problems
please indicate which ones you wish to have graded.

Policy on Misprints

The Ph.D. Comprehensive Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.