Ph.D. Real Analysis Qualifying Exam

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This exam contains 8 problems. Do 6 problems and if you do more than 6 indicate clearly which 6 problems you wish to be graded. To get full credit you must show all your work and state all the theorems you use.

1. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.
   (a) Give the definition for $f : \Omega \to \mathbb{R}$ to be $\mathcal{F}$-measurable.
   (b) Let $f, g : \Omega \to \mathbb{R}$ be two $\mathcal{F}$-measurable functions. Show that $\{x \in \Omega : f(x) = g(x)\}$ and $\{x \in \Omega : f(x) < g(x)\}$ are $\mathcal{F}$-measurable.
   (c) Let $\{S_j\}$ be a sequence of measurable sets such that
   $$\sum_{j=1}^{\infty} \mu(S_j) < \infty.$$
   Show that the set $\{x \in \Omega : x \in S_j \text{ for infinitely many values of } j\}$ is a measurable null set.

2. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space and $f \in L^1(\mu)$ is non-negative. Define $\lambda(A) = \int_A f d\mu$ for all $A \in \mathcal{F}$.
   (a) Show that λ is a measure.
   (b) Show that $\int_\Omega hf d\mu = \int_\Omega h d\lambda$ for all $h \in L^\infty(\mu)$.

3. Show that the measure space $(\Omega, \mathcal{F}, \mu)$ is σ-finite if and only if there exists $f \in L^1(\Omega, \mu)$ such that $f(x) > 0$ for all $x \in \Omega$.

4. Evaluate $\lim_{n\to\infty} \int_0^\infty \frac{n \sin(x/n)}{x + x^3} dx$. Justify your answer.

5. Let $\{f_j\}$ be a sequence of real valued Lebesgue measurable functions on $[0, 1]$. Assume that $f_j, f \in L^1([0, 1])$ for all j, $f_j \to f$ a.e. and $\|f_j\|_{L^1} \to \|f\|_{L^1}$ as $j \to \infty$. Show that $\|f_j - f\|_{L^1} \to 0$ as $j \to \infty$.

6. Prove or disprove the following statement: Let $\{f_n\}$ be a sequence of real valued continuous functions on $[0, \pi]$ such that $|f_n(x)| \le \sin x$ for all $0 \le x \le \pi$. Then $\{f_n\}$ has a subsequence which is uniformly convergent on $[0, \pi]$.

7. Prove or disprove the following statement: For every real valued, continuous function f on $[0, 1]$ such that $f(0) = 0$ and every $\varepsilon > 0$, there exists a real polynomial P having only ODD powers of x, that is P is of the form
   $$P(x) = a_1x^1 + a_3x^3 + \cdots + a_{2n+1}x^{2n+1},$$
   such that $\sup\{|f(x) - P(x)| : 0 \le x \le 1\} \le \varepsilon$.

8. Let $\{f_n\}$ be a sequence of real bounded linear functionals on a Banach space X. Show that either there exists $x \in X$ so that $f_n(x) \ne 0$ for all n or there exists n such that $f_n(x) = 0$ for all $x \in X$.