Ph.D. QUALIFYING EXAM
DIFFERENTIAL EQUATIONS
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Biao Ou and Henry Wente

This exam has two parts, ordinary differential equations and partial differential equations. Choose four problems from each part.

Part I: Ordinary Differential Equations

1. Consider the differential equation with initial condition

$$dx/dt = F(t,x), \quad x(a) = x_0 \in R^n$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ and
$F(t,x) = (F_1(t,x), F_2(t,x), \ldots, F_n(t,x))^T$. Suppose $F(t,x)$ is continuous for $a \le t \le b$ and $x \in R^n$ and satisfies a Lipschitz condition $|F(t,x) - F(t,y)| \le L|x-y|$ for $a \le t \le b$ and all x, y.

(a) Convert the differential equation with the initial condition into an equivalent integral equation.

(b) Set up the Picard iteration process and prove that the sequence converges uniformly on the interval [a, b] to a limit function $x_\infty(t)$.

(c) Show that $x_\infty(t)$ is a solution to the differential equation on [a, b].

(d) Establish that the solution to the differential equation with the given initial condition is unique.

2. Let $k(s)$ be a continuous function on [0,1]. Consider

$$\begin{cases} \frac{d^2x}{ds^2} = -k(s)\frac{dy}{ds}, & \frac{d^2y}{ds^2} = k(s)\frac{dx}{ds}; \\ x(0) = 0, \quad y(0) = 0, & x'(0) = 1, \quad y'(0) = 0. \end{cases}$$

1) Turn the problem into a system of first order linear differential equations with an initial condition.

2) Show that $x'(s)^2 + y'(s)^2 = 1$ for all s in [0,1].

3) Solve the system for the cases $k(s) = 0, k(s) = -2$ respectively.

3. Consider

$$\ddot{\theta} = -3\sin(\theta), \quad \theta(0) = \theta_0, \quad \dot{\theta}(0) = 0.$$

Show that the solution $\theta(t)$ is a periodic function if $0 < \theta_0 < \pi/2$. Find a formula for the period $T(\theta_0)$ and find the limits

$$\lim_{\theta_0 \to (\frac{\pi}{2})^-} T(\theta_0), \quad \lim_{\theta_0 \to 0^+} T(\theta_0).$$

4. Find all the eigenvalues and eigenfuntions of the Sturm-Liouville system

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0.$$

5. Solve the initial value problem.

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^t, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

6. Give an example of the initial value problem $y' = F(x, y), y(a) = c$ that has more than one solution.

Part II: Partial Differential Equations

1. (Dirichlet Problem on the Unit Disk) Let $f(\theta)$ be a continuous and 2π-periodic function with Fourier series

$$f(\theta) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\theta + b_k \sin k\theta).$$

Let

$$u(r,\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} r^k (a_k \cos k\theta + b_k \sin k\theta).$$

(a) Prove that the series for $u(r,\theta)$ converges uniformly on any disk $B_R = \{(r,\theta) \mid 0 \le r \le R\}$ with $R < 1$.

(b) Show how to rewrite the series for $u(r,\theta)$ in the form

$$u(r,\theta) = \int_0^{2\pi} f(\phi) P(r, \theta - \phi) d\phi$$

where P is the Poisson kernel satisfying

$$P(r,\phi) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\phi + r^2}.$$

(c) Prove that $\lim_{r \to 1^-} u(r,\theta) = f(\theta)$ uniformly.

2. (Removable Singularity for Harmonic Functions) Suppose $u(x,y)$ is a continuous harmonic function satisfying $|u(x,y)| \le M$ for some constant M on the deleted unit disk $0 < \sqrt{x^2+y^2} < 1$. Prove that the singularity at the origin is removable.

3. (Maximum Principle for the Heat Equation) Let

$$\Omega = \{(x,t) \mid x \in \omega, 0 < t < T\} \subset R^{n+1}$$

where ω is a bounded open set in R^n. Set

$$\begin{aligned} \partial'\Omega &= \{(x,t) \mid x \in \partial\omega, 0 \le t \le T \\ &\quad \text{or } x \in \omega, t = 0\} \end{aligned}$$

Let $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ satisfy $u_t - \Delta u \leq 0$ in Ω. Prove that

$$\max_{\bar{\Omega}} u = \max_{\partial'\Omega} u.$$

4. (Harnack's Inequality) Let $u(x, y)$ be a positive continuous harmonic function on the disk

$$B_a = \{(x, y) \mid x^2 + y^2 \leq a^2\}.$$

(a) Prove the Harnack Inequality

$$\frac{a-r}{a+r} u(0,0) \leq u(x,y) \leq \frac{a+r}{a-r} u(0,0)$$

where $r = \sqrt{x^2 + y^2} < a$.

(b) Show that if $u(x, y)$ is a positive continuous harmonic function on R^2 then $u(x, y)$ is a constant function.

5. Derive the complete solution to the one-dimensional wave equation $u_{tt} = c^2 u_{xx}$ on $-\infty < x < \infty, t > 0$ with the initial conditions

$$u(x, 0) = f(x) \quad u_t(x, 0) = g(x).$$

6. Consider the eikonal equation $u_x^2 + u_y^2 = u^2$.

(a) Find all solutions of the form $u(x, y) = f(x)$.

(b) Use (a) to write down a general solution $u = u(x, y, a, b)$. (Hint: Use the fact that the PDE is invariant under rotations in the xy plane.)

(c) Find the solution of the PDE satisfying the condition $u(x, x) = 3x$.