University of Toledo Algebra Ph.D. Qualifying Exam
January 27, 2007

Instructions: The exam is divided into three sections. Please choose exactly three problems from each section. Clearly indicate which three you would like graded. You have three hours.

1. Section I

(1) Let G be a group with exactly three subgroups (including the trivial subgroup and G itself).
  (a) Prove that G is finite and cyclic.
  (b) Prove that the order of G is p^2 for some prime p.

(2) Let G be a non-abelian p-group of order p^3, where p is a prime number. Let $Z(G)$ be the center of G and G' be its commutator subgroup.
  (a) Show that $Z(G) = G'$ and that this is the unique normal subgroup of G of order p.
  (b) Determine the number of distinct conjugacy classes of G.

(3) Let G be a finite group having exactly n Sylow p-subgroups for some prime p. Show that there exists a subgroup H of the symmetric group S_n of degree n that also has exactly n Sylow p-subgroups.

(4) Describe the isomorphism classes of groups of order 175 by giving a presentation of each with generators and relations.

(5) Let G be a finite group of order pqr for primes $p < q < r$. Prove that G is solvable.

2. Section II

(6) Prove or disprove: There exist two non-isomorphic rings, each with 9 elements, whose additive groups are isomorphic.

(7) Let $f(x) = x^5 - 9x + 3 \in \mathbb{Q}[x]$. Determine the Galois group of $f(x)$ over $\mathbb{Q}$. Hint: First use some basic calculus to prove that $f(x)$ has exactly 3 real roots and two complex (not real) roots.

(8) (a) Suppose that R is a commutative ring with identity. Prove that every maximal ideal of R is a prime ideal.
(b) Show that the ideal $(3, x)$ of $\mathbb{Z}[x]$ generated by 3 and x is a maximal ideal of $\mathbb{Z}[x]$.
(c) Find a prime ideal of $\mathbb{Z}[x]$ that is *not* maximal.

(9) Let K be the field obtained by adjoining to the rational numbers $\mathbb{Q}$ all complex cube roots of 2.
(a) Determine the degree $|K : \mathbb{Q}|$.
(b) Determine the Galois group of the extension $K/\mathbb{Q}$.
(c) Determine all subfields of K.

(10) Let α be a non-zero real number and suppose that $\alpha^n \in \mathbb{Q}$, the rational numbers, for some integer n. Let $g(x)$ be the minimal (monic) polynomial of α over $\mathbb{Q}$ and let $\deg g = m$.
(a) Show that $g(0) = \pm\alpha^m$.
(b) Prove that $g(x) = x^m - b$ for some $b \in \mathbb{Q}$.
(c) Show that m divides n.

3. Section III

(11) Let A be an $n \times n$ matrix over a field K and assume that the characteristic polynomial of A has distinct roots in the algebraic closure of K. Prove that any two $n \times n$ matrices that commute with A must commute with each other.

(12) Let V be a vector space over an algebraically closed field K and let $T : V \to V$ be a linear operator on V. Let $I : V \to V$ denote the identity operator. Show that V has a basis consisting of eigenvectors of T if and only if the kernel of $(\lambda I - T)^2$ is equal to the kernel of $\lambda I - T$ for all $\lambda \in K$.

(13) Let $\mathbb{Z}[i]$ denote the Gaussian integers. Prove or disprove:

$$\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$$

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(14) Let R be a commutative ring with 1 and let M be a left R-module. Prove that $\operatorname{Hom}_R(R, M)$ and M are isomorphic as left R-modules.

(15) Suppose R is a ring and $f : M \to M$ is an R-module homomorphism such that $f(f(m)) = f(m)$ for all $m \in M$. Prove:

$$M \cong \operatorname{Ker} f \oplus \operatorname{Im} f.$$