Ph.D. Qualifying Exam

Fall 2001

Instructions:

1. If you think that a problem is incorrectly stated ask the proctor. If her/his explanation is not to your satisfaction, interpret the problem as you see fit, but not so that the answer is trivial.

2. From each part solve 3 of the 5 five problems.

3. If you solve more than three problems from each part, indicate the problems that you wish to have graded.

Part A

1. Suppose that $\sum_{i=1}^{\infty} s_i$ is a series of positive terms with the property that $\sum_{i=1}^{n} s_i < \log(n)$. Show that $\sum_{i=1}^{\infty} \frac{s_i}{i}$ converges.

2. Consider a convergent sequence $\{a_n\}_{n=0}^{\infty}$ with $\lim_{n\to\infty} a_n = a$. Let

$$\alpha_n = \frac{a_0 + \cdots + a_n}{n},$$

and show that $\lim_{n\to\infty} \alpha_n = a$.

3. Denote the Banach space of absolutely convergent series of real numbers by ℓ^1. An absolutely convergent series $\sum_{i=1}^{\infty} a_i$ is said to converge at a rate determined by a positive convergent series $\sum_{i=1}^{\infty} b_i$ if there is a real number K such that for all i, $|a_i| < Kb_i$. Consider rates determined by positive series of the form $b_n = n^{-r}\log(n)^{-q}$ where r and q are rational with either $r > 1$ and q arbitrary, or $r = 1$ and $q > 1$. Let S be the set of all absolutely convergent series that do not converge at a rate determined by positive series of this form. Show that S is a dense subset of ℓ^1.

4. Suppose that $f(x)$ is a uniform limit of step functions defined on the closed interval $[a, b]$. Prove that at any point of $[a, b]$ the right and left limits of $f(x)$ exist.

5. (a) For any partition $\mathcal{P} = \{x_0, \dots, x_n\}$ of an interval $[a, b]$ and any $f(x)$ defined on $[a, b]$ define the variation of $f(x)$ relative to $\mathcal{P}$ by

$$V(f, \mathcal{P}) = \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)|$$

and define the total variation of $f(x)$ by $V(f) = \sup_{\mathcal{P}} V(f, \mathcal{P})$. The space of functions that have finite total variation is called the space of functions of bounded variation and is denoted by $BV([a, b])$. Show that if $f \in BV([a, b])$ then $||f|| = |f(0)| + V(f)$ defines a norm on $BV([a, b])$. (Correction: $||f|| = |f(a)| + V(f)$.)

(b) Show that if a subset of $BV([a, b])$ is open in the sup norm, then it is open in the norm defined in (a).

Part B

1. Let $f(x)$ be an integrable function on $\mathbf{R}$ and let $g(x)$ be the function defined by

$$g(x) = \int_0^1 tf(x + t)dt$$

Show that g is continuous on $\mathbf{R}$.

2. Either prove or give a counterexample to the following statement: given a sequence of measurable functions $\{f_n(x)\}$ defined on $[0, 1]$ converging pointwise to a limit $f(x)$ and a positive integrable function $g(x)$ on $[0, 1]$ such that $|f(x)| \leq g(x)$ for all $x \in [0, 1]$, then $f(x)$ is integrable and $\lim_{n \to \infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$.

3. Suppose the $\{f_c(x)\}_{c \in [a,b]}$ is a family of measurable functions defined on $\mathbf{R}$, and suppose that for each x, $c \to f_c(x)$ is continuous. Show that $g(x) = \sup\{f_c(x) | c \in [a, b]\}$ is measurable.

4. Suppose that $g \in L^\infty([a, b])$ and suppose that $\{f_n\}$ is a sequence of measurable functions converging to f in measure on $[a, b]$. Show that gf_n converges to gf in measure on $[a, b]$.

5. Suppose that $f(x)$ is a measurable function on $[0, \infty)$ with the property that $\int_0^\infty |f(x)|^2 dx < \infty$. Show that

$$\lim_{x \to \infty} x^{\frac{1}{2}} \int_x^\infty \frac{f(t)}{t} dt = 0$$

Prepared by: Zeljko Cuckovic, Geoffrey Martin