

Topology M.A. Comprehensive Exam

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This exam has been checked carefully for errors. If you find what you believe to be an error in a question, report this to the proctor. If the proctor's interpretation still seems unsatisfactory to you, you may alter the question so that in your view it is correctly stated, but not in such a way that it becomes trivial.

Do 4 of the following 7 problems.

1. Is $\{\frac{1}{n}\}_{n \in \mathbf{Z}_+}$ a closed subset of $[0, 1]$? Is it a closed subset of $(0, 1)$? Carefully justify your answer.
2. Suppose that X is metric space with metric $\delta: X \times X \rightarrow \mathbf{R}$. Let $c > 0$ be fixed, and define a function $\delta_c: X \times X \rightarrow \mathbf{R}$ by

$$\delta_c(x, y) = \begin{cases} \delta(x, y) & \delta(x, y) < c \\ c & \delta(x, y) \geq c. \end{cases}$$

Prove that δ_c is also a metric on X .

3. A function $f: X \rightarrow Y$ is defined to be continuous if for any open subset $U \subset Y$, $f^{-1}(U)$ is open in X . Suppose given a continuous function $f: X \rightarrow Y$. Show that if C is a closed subset of Y , then $f^{-1}(C)$ is a closed subset of X .
4. Suppose that X is an infinite set with the finite complement topology (*i.e.* a set is open if and only if its complement is finite). Show that X is not Hausdorff.

5. Consider the “deleted comb space,” which is the union of the following subsets of the plane:

$$\begin{aligned} & [0, 1] \times 0 \\ & 0 \times \{0, 1\} \\ & \frac{1}{n} \times [0, 1] \text{ for each } n \text{ a positive integer.} \end{aligned}$$

Show that the deleted comb space is connected but not path connected.

6. Suppose that X is a compact topological space and $f: X \rightarrow Y$ is continuous. Show that $f(X)$ is compact.
7. Suppose that X_1, \dots, X_n is a collection of finite sets, all with the discrete topology. Show that $X_1 \times \dots \times X_n$ has the discrete topology.