M.A. COMPREHENSIVE EXAM ALGEBRA JULY 17, 1999

Please give complete proofs. Do *two* problems from each of the three parts. If you do three problems in one of the parts please indicate which two problems you want graded.

PART I.

1. Let G be a finite abelian group and let z be the product of all of the elements in G. Prove that $z^2 = 1$. Give an example of $G \neq 1$ where $z = 1$ and another example where $z \neq 1$.
2. Let $n \geq 3$ and let H be the subgroup of the symmetric group S_n which is generated by the set of 3-cycles. Show that H is A_n, the alternating group on n letters.
3. Let G be a finite p-group and let H be a normal subgroup of G of order p. Show that H is contained in the center of G.

PART II.

4. Prove that $\mathbb{Z}[x]$, the polynomial ring over the integers $\mathbb{Z}$, is not a principal ideal domain.
5. Let R be a commutative ring with an identity element. Under addition R is an abelian group. Suppose that each subgroup of this group is an ideal of R. Show that the ring R is isomorphic to the ring of integers $\mathbb{Z}$, or to the integers modulo n, for some integer n.
6. Let R be a commutative ring with an identity element and let a be an element of R. Suppose that M is an ideal of R with the following two properties:
   (a) $a^n \notin M$ for $n = 1, 2, \cdots,$.
   (b) If K is an ideal of R which contains M but $K \neq M$, then $a^n \in K$ for some n. Prove that M is a prime ideal of R; that is, if $x, y \in R$ and $xy \in M$ then $x \in M$ or $y \in M$.

PART III.

7. Let A be a real symmetric $m \times m$ matrix and suppose that $A^n = I$ for some $n \geq 1$. Show that $A^2 = I$.

8. Let T and S be $n \times n$ matrices with entries from the field K. Suppose that N is the null-space of S (i.e., N is the set of $n \times 1$ column vectors annihilated by S). Show that $TN \subseteq N$.

9. Let T be an $n \times n$ matrix over the field K. Suppose that $p(x)$ is a nonzero polynomial of least degree with $p(T) = 0$. Show that T is invertible if and only if the constant term of $p(x)$ is $\neq 0$.