M.S. Comprehensive Exam in Analysis

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Solve any 6 of the following 8 problems.

1. Find the Laurent series for

$$f(z) = \frac{5z}{z^2 + z - 6} \quad \text{in the annulus } \ 1 < |z-1| < 4.$$

2. Use the Residue Theorem to evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}.$$

3. Let $u(x, y)$ and $v(x, y)$ be real-valued functions on a domain D.

   (a) If u and v are harmonic conjugates on D, is their product $u \cdot v$ harmonic on D?

   (b) If $u(x, y)$ is a nonconstant harmonic function on D, is u^2 harmonic on D?

4. Let f be analytic in the open unit disk $\{z \in \mathbf{C} : |z| < 1\}$. For each $0 < r < 1$, define $f_r(\theta) = f(re^{i\theta})$. Show that for all such r

$$f_r(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \mathbf{C}_r(\theta - t) \, dt$$

   where $\mathbf{C}_r(\theta) = \frac{1}{1-re^{i\theta}}$.

5. Construct a compact set of real numbers whose limit points form a countably infinite set. Prove that your set has the desired properties.

6. If $s_1 = \sqrt{2}$, and $s_{n+1} = \sqrt{2 + \sqrt{s_n}} \qquad (n = 1, 2, 3, \ldots)$, prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \ldots$.

7. For $n = 1, 2, 3, \ldots, x$ real, put

$$f_n(x) = \frac{x}{1 + nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a function f, and that the equation

$$f'(x) = \lim_{n \to \infty} f'_n(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

8. Suppose f is a bounded real function on $[a, b]$, and that f^2 is Riemann integrable on $[a, b]$. Does it follow that f is Riemann integrable on $[a, b]$? Prove or disprove.