Department of Mathematics
University of Toledo

Master of Science Degree
Comprehensive Examination
Probability and Statistical Theory

May 24, 1997

Instructions:

Do FOUR out of five questions.
Clearly state which four you are choosing.
Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
Books, notes, and calculators may be used.
This is a three hour test.
1. Consider an experiment in which balls numbered 1,...,n are distributed at random in n boxes so that the total number of outcomes is n!. Let \( S_n \) be the number of matches; i.e., the number of balls in boxes having the same number. Find \( E(S_n) \) and \( \text{Var}(S_n) \).

2. Let \( X_1,\ldots,X_n \) be a random sample from a uniform \( U(0,\theta) \) distribution, where \( \theta > 0 \).

   (a) Are \( X(n), \frac{X(n)}{X(n-1)}, \frac{X(n-1)}{X(n-2)}, \ldots, \frac{X(2)}{X(1)} \) independent? Explain your reasoning.

   (b) Find the UMVU estimator of \( \theta^2 \).

   (c) Let \( \theta = 1 \). Find the distribution of \( X_l - X_k \), where \( 1 \leq k < l \leq n \). In addition, find \( E(X_l - X_k) \) and \( \text{Var}(X_l - X_k) \).
3. Let \( X_1, \ldots, X_n \) denote a random sample from the distribution with density
\[
f(x; \alpha, \beta) = \alpha e^{\alpha x - \beta} \quad \text{for } x \leq \beta / \alpha \quad \text{where } \alpha > 0.
\]

a. Confirm that this is a density.
b. Find a sufficient statistic for \((\alpha, \beta)\).
c. Find the method of moments estimator (MME) for \((\alpha, \beta)\).
d. Find the maximum likelihood estimator for \(\alpha\) under the condition that \(\beta = 0\).
e. Find the maximum likelihood estimator for \((\alpha, \beta)\) with no restriction placed on \(\beta\).

(Hint: First show that for each fixed \(\alpha\), the maximum occurs along the line \(\beta = \alpha x(n)\), where \(x(n)\) denotes the maximum of the observations.)
f. Derive the likelihood ratio for testing \(H_0: \beta = 0\) versus \(H_1: \beta < 0\).
g. Assume that we have the following data from this distribution: \(n = 10\) and the ordered observations are \(-17, -9, -9, -7, -5, -4, -4, -3, -1, -1\). Find the MME and the MLE for the parameters \((\alpha, \beta)\). Also perform the large sample (chi-square approximation) likelihood ratio test derived in part f.

4. A lake has \(N\) fish, \(N\) unknown. We will find the MLE for estimating \(N\) in the context of the experiment described as follows:

i) Capture and tag 5 fish.
ii) Return these fish to the lake and stir for a while.
iii) Capture 7 fish and let \(X\) denote the number of tagged fish in the sample of size 7.

a. What is the distribution of \(X\)? Give the formula as a function of \(N\) and don't forget the range.
b. Based upon this one observation, give the method of moments estimator for \(N\).
c. Give the likelihood function, \(L(N; x)\), again for one observation.
d. Find the set of integers \(n\) for which \(L(n+1; x) \geq L(n; x)\), i.e., find \(\{n \geq 1 : L(n+1; x) \geq L(n; x)\}\).
e. Use your answer in (d) to give a formula for \(\hat{N}\), the MLE for \(N\).

5. Let \(X, Y\) be Uniformly distributed over \(A\) where \(A\) is the triangle with vertices at \((0, 0), (a, 0),\) and \((0, b)\).

a. Find the marginal distributions and the expectations of \(X\) and \(Y\).
b. Find the CDF and PDF of \(S = X + Y\).
c. Based on a sample of size \(n\) from this joint distribution, give the method of moments estimator of the parameter vector \((a, b)\).