Department of Mathematics
University of Toledo

Master of Science Degree
Comprehensive Examination
Probability and Statistical Theory

June 24, 1996

Instructions:
Do all four problems.
Show all of your computations in your Blue Book.
Prove all of your assertions or quote appropriate theorems.
Books, notes, and calculators may be used.
This is a three hour test.
1. (30) The probability density function of a continuous random variable $X$ is given by
\[
f(x) = \begin{cases} 
 cx & \text{if } 0 \leq x \leq a \\
 c(2a-x) & \text{if } a < x \leq 2a \\
 0 & \text{otherwise}
\end{cases}
\]
a. Find $c$.
b. Prove that $\mu = E(X) = a$.
c. Show that $\sigma = \text{StDev}(X) = a/\sqrt{6}$.
d. If $M(t)$ denotes the moment generating function of $X$, find $M(0)$, $M'(0)$, and $M''(0)$.

*** Parts e-g pertain to a random sample $X_1, X_2, \ldots, X_n$ from this distribution. Estimators should use all of the data!!! ***

e. Find the method of moments estimator of $a$.
f. Find two different unbiased estimators of $\sigma^2$.
g. Find an unbiased estimator of $\sigma$.

2. (20) Let $X_1, X_2, \ldots, X_n$ denote a random sample from the distribution with density given by
\[
f(x) = \frac{1}{\theta} e^{-x/\theta} \quad \text{for } x \geq 0.
\]
a. Find a complete and sufficient statistic for $\theta$.
b. Find the method of moments estimate for $\theta$.
c. Find the maximum likelihood estimate of $\theta^2$.
d. Let $Y = X_1/(X_1+X_2)$. Find $E(Y)$ and $\text{Var}(Y)$.

3. (35) Let $X_1, X_2, \ldots, X_n$ denote a random sample from the distribution with density given by
\[
f(x) = \frac{\alpha b^\alpha}{x^{\alpha+1}} \quad \text{for } x \geq b, \text{ where } \alpha > 0 \text{ and } b > 0.
\]
a. Confirm that this is a density.
b. Find a sufficient statistic for $(\alpha, b)$.
c. Find the method of moments estimates for $\alpha$ and $b$.
d. Find the maximum likelihood estimates of $\alpha$ and $b$.
e. Find the maximum likelihood estimate for $\alpha$ under the restriction that $b=1$.
f. Find the likelihood ratio test statistic for testing $H_0: b = 1$ versus $H_1: b > 1$.
g. Suppose that we have the following data ($n=20$) for performing this test (the top row is the actual observations with statistics at the end and the bottom row gives the natural logs with statistics):

| $x$ | 3.59 | 2.49 | 3.47 | 2.65 | 2.96 | 4.49 | 7.10 | 3.02 | 3.44 | 5.78 | 2.08 | 3.43 | 3.29 | 4.18 | 3.41 | 3.03 | 6.29 | 4.16 | 6.14 | 2.35 |
| $\ln(x)$ | 1.38 | 0.91 | 1.24 | 0.98 | 1.08 | 1.49 | 1.56 | 1.91 | 1.72 | 0.73 | 1.23 | 1.19 | 1.44 | 1.11 | 1.11 | 1.04 | 1.43 | 1.31 | 0.85 | 1.36 | 0.34 |

Perform the likelihood ratio test in part f using the usual large sample approximation. Set $\alpha = .05$.

4. (15) Let $(X,Y)$ be uniformly distributed over the triangle with vertices at $(0,0)$, $(0,a)$, and $(b,0)$. Find
a. the CDF and pdf of $\min(X,Y)$.
b. the method of moments estimators of $a$ and $b$ (based on a random sample from this joint distribution).