Department of Mathematics
University of Toledo

Master of Science Degree
Comprehensive Examination
Probability and Statistical Theory

May 11, 1996

Instructions:

You will do a total of five problems.

Do all four *Detailed Knowledge* Questions, numbered 1-4.

Select one from the *General Knowledge* Questions, numbered 5 and 6.

State whether you selected problem 5 or problem 6 on the cover of your Blue Book.

Show all of your computations in your Blue Book.

Prove all of your assertions or quote appropriate theorems.

Books, notes, and calculators *may be used.*

This is a three hour test.

1. [10 points] A deck of 52 cards is shuffled and split into two halves. Let X be the number of red cards in the first half.

(a) Find the probability frequency distribution of X.

(b) Find $E(X)$.

(c) Find $\text{Var}(X)$.

2. [20 points] Suppose that the random variables $X_1, \ldots, X_n$ are iid $N(\mu, \sigma^2)$, where $-\infty < \mu < \infty$ and $\sigma^2 > 0$.

(a) Find the UMVU estimate of σ.

(b) Suppose we want to test the null hypothesis $H_0 : \sigma^2 \leq \sigma_0^2$ versus the alternative hypothesis $H_A : \sigma^2 > \sigma_0^2$. Find the likelihood ratio test of size α.

3 . (10 points)

a. Derive the moment generating function for the U[0,1] (continuous uniform on the interval [0,1]) distribution. Be careful at t=0.

b. Use the answer in part (a) to find the expectation for U[0,1] distribution.

4 . (40 points, 5 points for each part) In this question, we explore in detail the likelihood ratio test of H_0: b-a=1 versus H_1:b-a>1 for data $X_1, \ldots, X_n$ which is a random sample from the U[a,b] (continuous uniform on the interval [a,b]) distribution. The problem is broken into many pieces and hopefully phrased in such a way that if you cannot do any one part, you should be able to use the information given to do the other parts.

a. For the given model and the parameter space $b-a \geq 1$, show that the maximum likelihood estimates of a and b are $\hat{a} = \min(X_i)$ and $\hat{b} = \max(X_i)$.

b. For the given model, under the null hypothesis that b-a = 1, show that if $\max(X_i) - \min(X_i) \leq 1$, then one choice for maximum likelihood estimates of a and b are $\hat{a} = \min(X_i)$ and $\hat{b} = \min(X_i) + 1$ and if $\max(X_i) - \min(X_i) > 1$, then the maximum likelihood estimates of a and b do not exist and, furthermore, the likelihood is identically 0 (as a function of a and b) in that case.

c. Show that the likelihood ratio test statistic is

$$\lambda = \begin{cases} 0 & \text{if } \max(X_i)-\min(X_i) > 1 \\ \{\max(X_i)-\min(X_i)\}^n & \text{if } \max(X_i)-\min(X_i) \leq 1 \end{cases}$$

d. Let $U = \min(X_i)$ and $V = \max(X_i)$. Show that under the null hypothesis, the joint pdf of (U,V) is given by $f(u,v) = n(n-1)(v-u)^{n-2}$ for $a \leq u \leq v \leq b = a+1$.

e. Let $D = \max(X_i) - \min(X_i)$. Show that the pdf of D is given by $f(d) = n(n-1)d^{n-2}(1-d)$ for $0 \leq d \leq 1$.

f . If the observed values for n=5 are 1.10, 1.27, 1.34, 1.45, and 1.62, use the likelihood ratio test to decide whether or not to reject the null hypothesis at the $\alpha = 0.05$ level of significance.

g. $W = -2\ln(\lambda)$ should have a large sample null distribution which is approximately chi-square with one degree of freedom. Show that the density of W converges to the chi-square(1) density as n tends to infinity.

h. Make a decision using the information in (f) using the chi-square approximation (even though n is only 5).

GENERAL KNOWLEDGE QUESTIONS (choose either 5 or 6 - 20 points)

5 . Say that we have an urn with R red balls, B blue balls, and G green balls. When n are selected at random without replacement, we let X denote the number of red balls selected, Y the number of blue balls selected, and Z the number of green balls selected.

a. Find the joint distribution of (X,Y,Z).

b. Find the mean and covariance matrix of (X,Y,Z).

c. Show that the covariance matrix is singular.

d. Find the equation the (two-dimensional) plane which contains the entire Range(X,Y,Z).

6. [20 points] Suppose $U_1, U_2, \ldots$ are independent uniform $(0,1)$ random variables, and let N be the first $n \geq 2$ such that $U_n > U_{n-1}$.

(a) Find $P(U_1 \leq u \text{ and } N = n)$ for $0 \leq u \leq 1$ and $n \geq 2$.

(b) Find $P(U_1 \leq u \text{ and } N \text{ is even})$ for $0 \leq u \leq 1$.

(c) Find $E(N)$.

Hint: Consider the events $\{N > n-1\}$ and $\{N > n\}$. What do these imply about $U_1, \ldots, U_{n-1}$ and $U_1, \ldots, U_n$, respectively?