Topology MA Comprehensive Exam

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This examination has been checked carefully for errors. If you find what you believe to be an error in a question, report this to the proctor. If the proctor's interpretation still seems unsatisfactory to you, you may modify the question so that in your view it is correctly stated, but not in such a way that it becomes trivial. If you feel that the examination is on the long side do not panic. The grading will be adjusted accordingly.

Do either eight questions from part one or six questions from part one and one question from part two or four questions from part one and two questions from part two. (Complete questions from part two are worth twice as much as complete questions in part one.) Indicate clearly which questions you want to be graded.)

1 Part One

1. Define what it means for (X, d) to be a metric space. If (X, d) is a metric space then $\{x \in X : d(x, x_0) \leq \epsilon\}$ is said to be the *closed ball* of radius ϵ and center x_0. Prove that a closed ball is a closed set.

2. A set is said to have the finite complement topology if the closed sets are the finite sets together with the entire set. Let $f : \mathbb{R} \to \mathbb{R}$ be the identity map $f(x) = x$ where in the domain $\mathbb{R}$ has the usual topology but in the codomain it has the finite complement topology. Show that f is continuous. Is f a homeomorphism? Explain your answer.

3. Define the term *closure* $\overline{A}$ of a subspace A of a topological space X. Prove that if A and B are subspaces of X and B is closed and $A \subset B$ then $\overline{A} \subset B$.

4. Let X be a topological space. The diagonal of $X \times X$ is the subset $\Delta = \{(x, x) : x \in X\} \subset X \times X$. Show that X is Hausdorff if and only if Δ is closed in $X \times X$ using the product topology.

5. Let $X = \Pi_{\mu \in M} X_\mu$ be the Cartesian product of the topological spaces $(X_\mu)_{\mu \in M}$ and let X have the product topology. Recall that a space is T_1 if any two distinct points in the space can be separated by not necessarily disjoint open sets. Show that if each $(X_\mu)_{\mu \in M}$ is T_1 then so is X. If you cannot do it for T_1 spaces do it for Hausdorff spaces.

6. Prove that the continuous image of a connected set is connected. Prove that a path-connected topological space is connected.

7. *Assume* that every completely regular space is homeomorphic to a subspace of a power of $[0, 1]$. Show that a topological space X is compact Hausdorff if and only if X is homeomorphic to a *closed* subspace of a power of $[0, 1]$.

8. Show that if A and B are compact subspaces of a topological space X then $A \cup B$ is compact. If in addition X is Hausdorff, show that $A \cap B$ is compact.

9. Define the terms *filter* and *ultrafilter*. Show that every filter is equal to the intersection of all ultrafilters that contain it.

10. Let $(X_i)_{i \in I}$ be a family of compact topological spaces and $X = \Pi_{i \in I} X_i$ (endowed with the product topology). Assume that a topological space is compact if and only if every ultrafilter converges. Show that X is compact. (Note: You are asked to *prove* the famous *Tychonoff Theorem*.)

2 Part Two

1. Let $\mathbb{Q}$ denote the rational numbers as a subspace of $\mathbb{R}$ with the usual topology. Show that:

   (i) $\mathbb{Q}$ is not compact.

   (ii) $\mathbb{Q}$ is not locally compact.

   (iii) $\mathbb{Q}$ is not connected.

   (iv) $\mathbb{Q}$ is totally disconnected.

   (v) $\mathbb{Q}$ is zero-dimensional.

2. (i) Let D_2 be the 2-point space $\{0, 1\}$ with the discrete topology and consider the product space $X = D_2^{\mathbb{N}}$. Show that X is a Hausdorff space.

   (ii) Show that X is a compact space.

   (iii) Show that X has no isolated points, that is, no singleton is open in X.

   (iv) Show that X is zero-dimensional, that is, X has a basis of clopen sets.

   (v) Show that X is second countable, that is, X has a countable basis.

   (Note: A non-empty topological space satisfying these five properties is homeomorphic to the *Cantor set*.)