M.S and M.A Comprehensive Analysis Exam

Željko Čučković and Sönmez Şahutoğlu

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To get full credit you must show all your work.
This exam contains 6 real analysis and 6 complex variables questions.

Real Analysis

100% will be obtained for complete answers to four questions. Indicate clearly which four questions you wish to be graded.

1. (a) Define the supremum of a bounded set $A \subset \mathbb{R}$.
   (b) Let A and B be two bounded sets in $\mathbb{R}$. Show that $\sup(A+B) = \sup A + \sup B$.
2. (a) Define a convergent sequence in $\mathbb{R}$.
   (b) Define a Cauchy sequence in $\mathbb{R}$.
   (c) Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}$. Show that $\{x_n\}$ is convergent if and only if it has a convergent subsequence.
3. (a) Define uniformly continuous functions on $\mathbb{R}$.
   (b) Give an example of a continuous function $f : (0,1) \to \mathbb{R}$ that is not uniformly continuous.
   (c) Let $f : (0,1) \to \mathbb{R}$ be a bounded continuous function. Show that $g(x) = x(x-1)f(x)$ is uniformly continuous on $(0,1)$.
4. Use the definition of Riemann integration to find the $\int_0^1 (x-1)dx$.
5. Let $X = \{a, b, c\}$ be a set of three points and $(\alpha, \beta) \in \mathbb{R}^2$. Define $d_{\alpha\beta}$ as a function on $X \times X$ so that $d_{\alpha\beta}(a,b) = 1, d_{\alpha\beta}(b,c) = \alpha, d_{\alpha\beta}(c,a) = \beta$. Find all possible (α, β) so that $d_{\alpha\beta}$ is a metric.
6. (a) Define compact subset of a metric space.
   (b) Use the definition of compactness to show that the interval $(0,1) \subset \mathbb{R}$ is not compact.
   (c) Let (X, d) be a compact metric space. Use the definition of compactness to show that every bounded sequence in X has a subsequence convergent in X.

Complex Analysis

100% will be obtained for complete answers to four questions. Indicate clearly which four questions you wish to be graded.

1. Find an analytic mapping that maps the domain $\{z \in \mathbb{C} : -\pi < \mathrm{Im}(z) < 3\pi\}$ onto the lower half plane $\{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$.

2. Let $C_\rho = \{z \in \mathbb{C} : |z| = \rho\}$ for $0 < \rho < 1$ oriented in the counterclockwise direction. Suppose that $f(z)$ is continuous on the disk $\{z \in \mathbb{C} : |z| < 1\}$. Show that

$$\lim_{\rho \to 0^+} \int_{C_\rho} \frac{f(z)}{z^{\frac{1}{2}}} dz = 0$$

where $z^{\frac{1}{2}}$ is the principal branch.

3. Expand $f(z) = \dfrac{z}{(z+1)(z-1)^2}$ in a Laurent series valid for $\{z \in \mathbb{C} : 0 < |z-1| < 2\}$.

4. Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^4} dx$.

5. (a) State the Maximum modulus principle for analytic functions.

   (b) Let $f(z) = \dfrac{z^4}{z^2+10}$. Find the maximum value of $|f(z)|$ on $\{z \in \mathbb{C} : |z| \leq 2\}$.

6. Find an entire function whose imaginary part is $v(x, y) = x^2 - y^2 + 2$.