MA exam: Algebra — 20 April 2013

Please do four problems, including one from each of the three sections. Give complete proofs — do more than simply quote a theorem. Please indicate clearly which four problems you want to be graded.

Part I: Group theory

1. Let G be an arbitrary group (not necessarily finite) and let $p > 0$ be a prime. Suppose $x \in G$ is an element of finite order $n = p^k m$ where m is prime to p (i.e., m is not divisible by p).

- **(a)** Show that $x = yz = zy$ for some $y, z \in G$, where y and z both have finite order, the order of y is a power of p, and the order of z is prime to p.
- **(b)** Keeping the notation of part (a), suppose that $p = 2$, $k = 3$, and $m = 15$ (so $n = 2^3 15 = 120$). Find a pair of y and z as guaranteed in part (a), expressing them as powers of x.

2. Let G be a finite group and assume that H and K are subgroups of G such that the product of the orders of H and K is strictly greater than the order of G.

- **(a)** Prove that $H \cap K \neq \{1\}$ where $1 \in G$ is the identity element.
- **(b)** Now suppose that K is a normal subgroup of G. What is the smallest possible order (in terms of the orders of H, K, and G) that $H \cap K$ could have?

Part II: Ring theory

3. An element a of a ring is said to be *nilpotent* if $a^k = 0$ for some positive integer k.

- **(a)** Suppose that $n > 1$ is an integer and that every element of the ring $\mathbb{Z}/n\mathbb{Z}$ is either a unit or a nilpotent element. Prove that $n = p^m$ for some prime p and positive integer m.
- **(b)** If p is a prime and m is a fixed positive integer, does the ring $\mathbb{Z}/p^m\mathbb{Z}$ have more units or more nilpotent elements? How many of each?

4. Let $g(x) = x^3+3x+2 \in F[x]$, where $F = \mathbb{Z}/7\mathbb{Z}$ is the finite field with seven elements, and let $I = (g(x))$ be the ideal of $F[x]$ generated by $g(x)$. Let $K = F[x]/(g(x))$ and let $\alpha = x + I \in K = F[x]/(g(x))$.

- **(a)** Prove that K is a field that contains a subfield isomorphic to F and a root of $g(x)$.
- **(b)** Find a polynomial $p(x) \in F[x]$ such that $\alpha^{-1} = p(\alpha)$.

Part III: Linear algebra

5.
- **(a)** Let V and W be vector spaces and let T be a linear operator from V into W. Suppose that V is finite-dimensional. Prove $\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V)$.
- **(b)** Let S be the linear operator defined on the space of 3×3 real matrices given by,

$$S(A) = A + A^t,$$

where A^t denotes the transpose of the matrix A. Determine the rank of S.

6. Let A be the 4×4 real matrix

$$\begin{pmatrix} 0 & 4 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

- **(a)** Find the characteristic polynomial of A and the eigenvalues of A.
- **(b)** Find a basis for each eigenspace of A.
- **(c)** Find J, the Jordan canonical form of A.
- **(d)** Find an invertible P such that $AP = PJ$.