MS COMPREHENSIVE EXAM
DIFFERENTIAL EQUATIONS
SPRING 2012

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This exam has two parts, (A) ordinary differential equations and (B) partial differential equations. Do any three of the four problems in each part. Clearly indicate which three problems in each part are to be graded. Show the details of your work.

Part A: Ordinary Differential Equations

1. Consider the second order linear homogeneous ordinary differential equation

$$(1-x^2)y'' - xy' + 4y = 0.$$

Assume a solution of the form

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

to find two linearly independent solutions.

2. Let y_1 and y_2 be two differentiable functions with second derivatives defined on (a,b). Show that the Wronskian

$$W(y_1,y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

is nonzero where t is in (a,b) if y_1 and y_2 are linearly independent on interval (a,b) and if y_1 and y_2 are solutions to $y'' + p(t)y' + q(t)y = 0$ where p and q are continuous on (a,b).

3. Let $A = \begin{pmatrix} 1 & 1 \\ 9 & 1 \end{pmatrix}$.

   (a) Find all eigenvalues of A.

   (b) For each eigenvalue of A, find all corresponding eigenvectors.

   (c) Find the general solution to $x' = Ax$.

4. (a) Find all critical points of

$$\begin{cases} x' = -x + y - x(y - x) \\ y' = -x - y + 2x^2 y \end{cases} .$$

   (b) Classify the critical point $x = 0$, $y = 0$ as to the type and stability. Refer to the attached table 9.3.1. Provide a phase plane portrait.

You may work completely any three of the four problems.

Part B: Partial Differential Equations

1. Solve explicitly for $u(x,y)$:

$$u(xu_x - yu_y) = y^2 - x^2.$$

2. Show that $z(x,y) = f[u(x,y)]$ satisfies the equation $a(x,y)z_x + b(x,y)z_y = 0$.

3. The Cauchy-Riemann equations

$$\begin{cases} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \end{cases}$$

   are a system of linear PDE's. Write the system in matrix form and obtain the characteristic equation.

4. Given the <u>two</u> initial-value problems

$$\begin{aligned} u_{xx} - c^{-2}u_{tt} &= 0 & -\infty < x < \infty \\ u(x,0) &= f(x) & 0 < t < \infty \\ u_t(x,0) &= 0 & \end{aligned}$$

   and

$$\begin{aligned} v_{xx} - c^{-2}v_{tt} &= 0 & -\infty < x < \infty \\ v(x,0) &= 0 & 0 < t < \infty \\ v_t(x,0) &= g(x). & \end{aligned}$$

   Let $w(x,t) = u(x,t) + v(x,t)$ and formulate a new IVP in $w(x,t)$.