M.S and M.A Comprehensive Analysis Exam

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To get full credit you must show all your work.
This exam contains 6 real analysis and 6 complex variables questions.

Real Analysis

100% will be obtained for complete answers to **four** questions out of six. Indicate clearly which four questions you wish to be graded.

1. Let $\{f_n\}$ be a sequence of continuous real-valued functions on the closed interval $[a, b]$.
   (a) Define what it means for f_n to converge uniformly to a function f on $[a, b]$ and prove that if f_n converges uniformly to f, then the limit function f is continuous.
   (b) Give an example of a sequence of continuous functions $\{f_n\}$ on $[0, 1]$ which converges pointwise to a function f on $[0, 1]$ and which is not continuous there.

2. Let f be a continuous real-valued function on $[a, b]$ and suppose that

$$\int_a^b x^n f(x)\, dx = 0, \quad \forall n \geq 0.$$

   Show that $f(x)$ is identically zero.

3. Let $E \subset X$ be a compact subset in the metric space (X, d). For $p \in X$ define

$$d(p, E) := \inf\{d(p, q) | q \in E\}.$$

   Show that for each fixed p, there exists a $q^* \in E$ with $d(p, E) = d(p, q^*)$.

4. Let f be a continuous real-valued function on $[a, b]$. Suppose $f(x) \geq 0$ and that there is an $x_0 \in [a, b]$ with $f(x_0) > 0$. Show that

$$\int_a^b f(x)\, dx > 0.$$

5. Let f be a continuous real-valued function on $[a, b]$. Show that f is uniformly continuous there.

6. Suppose the sequence $\{f_n\}$ is converging uniformly to f on $[a, b]$, where f_n and f are continuous.

   - Prove that
   $$\int_a^b f(x)\, dx = \lim_{n \to +\infty} \int_a^b f_n(x)\, dx.$$
   What about
   $$\lim_{n \to +\infty} \int_a^b \sin(f_n(x))\, dx \ \ ?$$
   - Give an example of continuous functions $\{f_n\}$ and f where f_n converges pointwise to f on $[0, 1]$ and yet
   $$\int_0^1 f(x)\, dx \neq \lim_{n \to +\infty} \int_0^1 f_n(x)\, dx.$$

Complex Analysis

100% will be obtained for complete answers to **four** questions out of six. Indicate clearly which four questions you wish to be graded.

1. Given the function
$$f(z) = \frac{z}{(z-2)(z+i)}$$
determine where it is analytic and where its singularities are located. Determine also the type of singularities. Finally expand it in a Laurent series in the following regions:
   - $|z| < 1$;
   - $1 < |z| < 2$.

2. State and prove Cauchy's integral formula.

3. Evaluate the following integral, where C is a simple closed curve going around the origin counterclockwise:

$$\oint_C \frac{e^{z^2}}{3z^2}\, dz$$

4. The complex logarithmic function:

   - Define $\log(z)$;
   - Use the definition and Cauchy-Riemann equations to prove that $\log(z)$ is an analytic function where it is defined.
   - Show that is multi-valued, and compute all determinations of $\log(2i)$.

5. Let C be an open upper semicircle of radius R with its center at the origin, and suppose $R > a > 0$.

   - Let $f(z) = \frac{1}{z^2+a^2}$. Show that **on** C

   $$|f(z)| \leq \frac{1}{R^2 - a^2}$$

   and that

   $$\left| \int_C f(z)\, dz \right| \leq \frac{\pi R}{R^2 - a^2}.$$

   - Use the previous result and Residue Theorem to compute

   $$\int_{-\infty}^{\infty} \frac{1}{x^2 + a^2}.$$

6. Find the location of branch points and discuss branch cuts for the following function:

$$f(z) = ((z-1)(z+1))^{\frac{1}{2}}.$$