Part A: Ordinary Differential Equations

1. Consider the initial value problem

\[ u'' - (1 - u^2)u' + u = 0, \quad u(0) = 2, \quad u'(0) = 0. \]

(a) Convert the initial value problem given above to a first order system of ordinary differential equations with initial conditions.

(b) Apply one step of Euler's numerical method for vectors with stepsize \( h = 0.01 \) to the first order system with initial conditions found in part (a) above.

(c) Use the results of part (b) above to estimate the values of \( u(0.01) \) and \( u'(0.01) \) in the original initial value problem.

2. Consider the second order equation

\[ y'' + p(x)y' + q(x)y = 0 \]

on an open interval \( I \) where \( p \) and \( q \) are continuous.

(a) State the Bolzano-Weierstrass Theorem.

(b) Let \( y \) be a non-zero solution. Prove that all zeros of \( y \) are isolated.

(c) Let \( y_1 \) and \( y_2 \) be two linearly independent solutions. Prove that between any two consecutive zeros of \( y_1 \) there is exactly one zero of \( y_2 \).
Part B: Partial Differential Equations

1. Suppose that $g(t) = \int_0^t f(\tau) \, d\tau$. If $G(s)$ and $F(s)$ are the Laplace transforms of $g(t)$ and $f(t)$ respectively, show that

$$G(s) = \frac{1}{s} F(s).$$

2. Find the general solution $u(x, y, z)$ for the equation

$$yu_x - xu_y + yz u_z = x$$

by using the method of characteristics.

3. Given the heat equation

$$u_t = u_{xx},$$

assume $u = f \left[ \frac{x}{\sqrt{t}} \right]$ and determine the function $f(x, t)$.

4. Use separation of variables to find three general solutions (depending on the separation constant values) to Laplace's equation:

$$u_{xx} + u_{yy} = 0.$$