Topology MA Comprehensive Exam

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This examination has been checked carefully for errors. If you find what you believe to be an error in a question, report this to the proctor. If the proctor's interpretation still seems unsatisfactory to you, you may modify the question so that in your view it is correctly stated, but not in such a way that it becomes trivial. If you feel that the examination is on the long side do not panic. The grading will be adjusted accordingly. Please write as legibly as you can: the examination may need to be scanned for grading.

1 Do six questions: Time 2 hours

1. Let $f : A \mapsto Y$ be a map of topological spaces with f being continuous and Y Hausdorff. Suppose that $A \subset X$ where X is a topological space. Prove that f has at most one continuous extension to a map from the closure $\overline{A}$ to Y.

2. Define the term *identification map* in the category of topological spaces. Let $\pi : X \to Y$ be a surjective, continuous map of topological spaces. Suppose that π maps closed sets to closed sets. Show that π is an identification map. What happens if we replace closed sets by open sets? Justify your answers.

3. Prove that a topological space X is Hausdorff if and only if the diagonal := $\{(x, x)$ such that $x \in X\}$ is closed in the product $X \times X$.

4. Prove that the closed interval $[0, 1]$ considered as a subset of $\mathbb{R}$ in the usual topology is compact.

5. Let G be a topological group that acts on a topological space X. Prove that if both G and the quotient space are X/G connected then X is connected.

6. A topological space X is said to be *locally connected* if the connected components of each point form a base of neighborhoods of X. Prove that in a locally connected space the connected components of X are both closed and open in X.

7. Give the definition of continuity for a function from one topological space to another. *Use your definition* to show that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is not continuous.

8. Let $\Phi = \Pi_{\mu \in M} X_\mu$ be the product of the topological spaces $(X_\mu)_{\mu \in M}$ and with Φ having the product topology. Prove that the projection $p_\mu : \Phi \to X_\mu$ is an open map from Φ onto X_μ for each $\mu \in M$.

9. Let γ be a given cover of a topological space X. Let us assume that, for each member $A \in \gamma$, there is given a continuous map $f_A : A \to Y$ such that

$$f_A \mid A \cap B = f_B \mid A \cap B$$

for each pair of members A and B of γ. Then we may define a function $f : X \to Y$ by taking

$$f(x) = f_A(x), \qquad (\text{if } x \in A \in \gamma).$$

Prove that if γ is a finite closed cover of X, then the function f is continuous.

10. Prove that every continuous image of a connected set is connected.

11. Prove that if two connected sets A and B in a space X have a common point p, then $A \cup B$ is connected.

12. Prove that every compact set K in a Hausdorff space X is closed.