1 Do eight questions

1. If \((X, d)\) is a metric space then \(\{x \in X : d(x, x_0) \leq \epsilon\}\) is said to be the closed ball of radius \(\epsilon\) and center \(x_0\). Prove that a closed ball is a closed set.

2. Define what it means for \((X, d)\) to be a metric space. Then \(d : X \times X \to \mathbb{R}\): is \(d\) continuous? Discuss. If you cannot answer in general do it for \(X = \mathbb{R}\) with the usual topology.

3. Prove or disprove: if a metric space is compact then it is bounded.

4. A set is said to have the finite complement topology if the closed sets are the finite sets together with the empty set. Let \(f : \mathbb{R} \to \mathbb{R}\) be the identity map \(f(x) = x\) where in the domain \(\mathbb{R}\) has the usual topology but in the codomain it has the finite complement topology. Show that \(f\) is continuous. Is \(f\) a homeomorphism? Explain your answer.

5. Let \(B\) be an open subset of a topological space \(X\). Prove that a subset \(A \subset B\) is relatively open in \(B\) if and only if \(A\) is open in \(X\).

6. Define the term closure \(\overline{A}\) of a subspace \(A\) of a topological space \(X\). Prove that if \(A\) and \(B\) are subspaces of \(X\) and \(B\) is closed and \(A \subset B\) then \(\overline{A} \subset B\).

7. Let \(X = \prod_{\mu \in M} X_\mu\) be the Cartesian product of the topological spaces \((X_\mu)_{\mu \in M}\) and let \(X\) have the product topology. Recall that a space is \(T_1\) if any two distinct points in the space can be separated by not necessarily disjoint open sets. Show that if each \((X_\mu)_{\mu \in M}\) is \(T_1\) then so is \(X\). If you cannot do it for \(T_1\) spaces do it for Hausdorff spaces.

8. Define compactness for a topological space. A collection of subsets is said to have the finite intersection property if every finite class of those subsets has non-empty intersection. Prove that a topological space is compact iff every collection of closed subsets that has the finite intersection property itself has non-empty intersection.
9. Prove that the continuous image of a connected set is connected. Prove that a path-connected topological space is connected.

10. It is a fact that every compact subset of a Hausdorff space is closed. Moreover a topological space is said to be normal if every pair of disjoint closed sets can be separated by disjoint open sets. Prove that a compact Hausdorff space is normal.

11. Let $X$ be a topological space. Let $A \subset X$ be connected. Prove that the closure $\overline{A}$ of $A$ is connected.

12. Define the term “identification map.” If $f$ maps open sets to open sets and is surjective show that $f$ is an identification map. What happens if we replace “open” by “closed” in the preceding sentence?

13. Prove or disprove: in a compact topological space every infinite set has a limit point. If you cannot answer the question for a compact topological space answer it for a metric space.

14. Prove that $\mathbb{R}$ is connected.