MS COMPREHENSIVE EXAM
DIFFERENTIAL EQUATIONS
SPRING 2007

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This exam has two parts, (A) ordinary differential equations and (B) partial differential equations. Choose three problems in part (A) and four problems in part (B). Mark clearly the problems you choose and show the details of your work.

Part A: Ordinary Differential Equations

1. Find the general solution to the system

$$\begin{cases} \dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = 9x_1 + x_2 \end{cases}$$

2. Consider the second order equation

$$y'' + p(x)y' + q(x)y = 0$$

on an open interval I where p and q are continuous.

   (a) State the Bolzano-Weierstrass Theorem.

   (b) Let y be a non-zero solution. Prove that all zeros of y are isolated.

   (c) Let y_1 and y_2 be two linearly independent solutions. Prove that between any two consecutive zeros of y_1 there is exactly one zero of y_2.

3. Consider the initial value problem

$$\frac{d^2y}{dx^2} - (1-y^2)\frac{dy}{dx} + y = 0$$

$$\begin{cases} y(0) = 2 \\ \frac{dy}{dx}(0) = 1. \end{cases}$$

(a) Convert the above initial value problem to a first order system of ordinary differential equations with initial conditions by letting $y_1 = y$ and $y_2 = \frac{dy}{dt}$.

(b) Apply Euler's numerical method in vector form with stepsize $h = .01$ to the initial value problem described in problem 3(a) above by calculating the approximations to $y(.01)$ and $\frac{dy}{dt}(.01)$.

4. Consider the nonlinear system

$$\begin{aligned} x' &= -x + y - x(y-x) \\ y' &= -x - y + 2x^2y \end{aligned}$$

(a) Find all critical points of the nonlinear system.

(b) Write the linear part as a system of ordinary differential equations.

(c) Determine whether the critical point at $x = 0$ and $y = 0$ is asymptotically stable, stable but not asymptotically stable or unstable.

(d) Provide a sketch in the phase plane of the trajectories in a neighborhood of $x = 0$ and $y = 0$.

Part B: Partial Differential Equations

1. Find the general solution $u(x, y)$ to

$$u_{xxy} = 1.$$

2. Given the solution $u(x, t) = f(x^2 + t^2)$ for f an arbitrary function; find a partial differential equation for $u(x, t)$.

3. The polynomials $p_0(x) = a$, $p_1(x) = bx + c$, $p_2(x) = dx^2 + ex + f$ (for a, b, c, d, e, f constants) are an orthogonal set on $[0, 1]$ and $p_2(0) = 1$. Find $p_2(x)$.

4. Solve the problem:

$$\begin{aligned} u_t &= ku_{xx} && t > 0,\ 0 < x < L \\ u_x(0,t) = 0,&\ u_x(L,t) = 0 && t > 0 \\ u(x,0) &= f(x) && 0 < x < L \end{aligned}$$

5. Find a general solution by separation of variables to

$$u_x + 2u_y - u = 0.$$