Master of Science Comprehensive Analysis Exam
Spring 2007
Real and Complex Analysis

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To obtain full credit you must show all your work

Part 1. Real Analysis. 100% will be obtained for *complete* answers to four questions. Indicate clearly which four questions you wish to be graded.

1. Let $x_1 = 1$ and $x_{n+1} = \dfrac{x_n + 1}{3}$ for $n \geq 1$.

   (a) Use induction to show that $x_n > \dfrac{1}{2}$ for all n.

   (b) Show that the sequence $\{x_n\}$ is decreasing.

   (c) Show that the limit of $\{x_n\}$ exists and find the limit.

2. Use the definition of Cauchy sequence to prove that if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences then $\{2x_n + y_n\}$ is a Cauchy sequence.

3. Suppose that the function $f : R \to R$ has limit L at 0, and let $a > 0$. If $g : R \to R$ is defined by $g(x) = f(ax)$ for $x \in R$, show that $\lim_{x \to 0} g(x) = L$.

4. Use the definition to show that if $f(x)$ is Riemann integrable on $[a, b]$ and $f(x)$ is Riemann integrable on $[b, c]$ then $f(x)$ is Riemann integrable on $[a, c]$.

5. (a) In each of the following, determine if the given series converges. Explain your answer.

(1) $\sum_{n=1}^{\infty} \frac{n^4}{n!}$

(2) $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^4 - 2}}$

(b) Find the radius of convergence and the interval of convergence for the following series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n2^n} x^n$$

Part 2. Complex Analysis. 100% will be obtained for *complete* answers to four questions. Indicate clearly which four questions you wish to be graded.

1. Evaluate the Cauchy principal value of

$$\int_0^{\infty} \frac{x \sin x}{x^2 + 9}\, dx$$

2. Expand $f(z) = \dfrac{1}{(z-1)(z-3)^2}$ in a Laurent series valid for:

(a) $0 < |z - 1| < 2$

(b) $0 < |z - 3| < 2$

3. Find the image of the following curves under the reciprocal mapping $w = \frac{1}{z}$. Draw the graphs.

   (a) the semicircle $|z| = 2$, $0 \leq \arg z \leq \pi$

   (b) the line $x = 1$

4. Find where the following functions are analytic.

   (a) $f(z) = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$

   (b) $f(z) = 3x^2y^2 - 6ix^2y^2$

5. Evaluate $\int_C \frac{dz}{z^2 + 1}$ where C is the circle $|z| = 4$.